Prerequisites for Math 212: Differential Equations

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Introduction

These notes are intended to briefly recall most of the prerequisites for a course on differential equation based on the textbook *Elementary Differential Equations*, 10th edition, by William E. Boyce and Richard C. DiPrima, held in the Spring and Fall semester at Emory university.

These notes briefly go over some special functions, solving small systems of linear euations, techniques to find the roots of quadratic, cubic and biquadratic polynomials, and basic rules and techniques of differentiation and integration. Wherever possible, proofs and lengthy explanations will be avoided, but to most of the covered topics, exercises are provided.

These notes are intended as an additional resource for students, they do not aim to replace the mandatory calculus courses.

Should there be any typos or mathematical errors in this manuscript, I'd be glad to hear about them via email (`michael.mertens@emory.edu`) and correct them.

Atlanta, August 2015, Michael H. Mertens

Contents

1 Special functions

1.1 The exponential function and the logarithm

Probably one of the most important functions in mathematics is the *exponential function*, often denoted either by

$$\exp(x) \quad \text{or} \quad e^x.$$

The number $e = \exp(1)$ is known as *Euler's number* and has the numerical value

$$e = 2.71828182845904523536028747\ldots .$$

The exponential function is usually defined by the following infinite series,

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

where $k!$ (read "k factorial") is the product of all natural numbers up to k and we set $0! := 1$. It can be used to build several other special functions, such as the trigonometric functions and the hyperbolic functions. We collect some of the most important properties of the exponential function in the following theorem.

Theorem 1.1. *The exponential function has the following properties.*

1. $\exp(x) > 0$ *for all real numbers* x*. In particular,* $\exp(x)$ *is never zero.*
2. *The exponential function is strictly monotonically increasing, i.e., for* $x < y$ *we have* $\exp(x) < \exp(y)$*.*
3. *We have the functional equations* $\exp(x) \cdot \exp(y) = \exp(x + y)$ *for all real numbers* x *and* y*.*
4. *The exponential function is its own derivative (see Section 4), i.e., we have* $\exp'(x) = \exp(x)$*.*

Another important fact about the exponential function is that it grows faster than any power function.

Proposition 1.2. *Let* a *be any real number, then we have*

$$\lim_{x\to\infty} \frac{x^a}{\exp(x)} = 0.$$

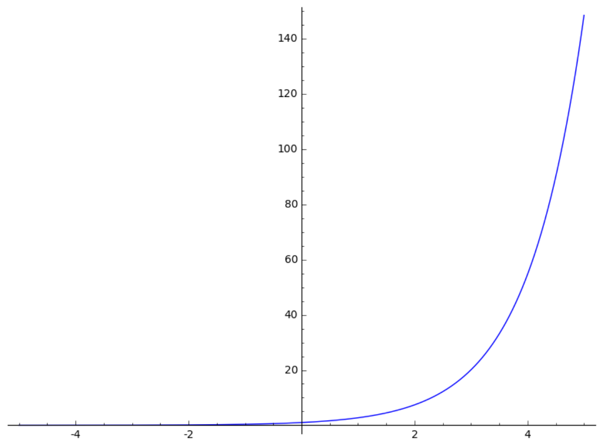

Figure 1.1: The exponential function

Here is a plot of the exponential function.

Since it is strictly increasing, the exponential function has an inverse function, the *natural logarithm*, which is denoted by $\log(x)$ or $\ln(x)$. This means that for all real x we have $\log(\exp(x)) = x$ and for all positive x we have $\exp(\log(x)) = x$. The logarithm is only defined for positive numbers because $\exp(x)$ assumes only positive values. It inherites its properties form the exponential function.

Theorem 1.3. *The logarithm has the following properties.*

1. *The expression* $\log(x)$ *is only defined for* $x > 0$ *and for every real* y*, there is an* $x > 0$ *such that* $\log(x) = y$.

2. *The logarithm is strictly monotonically increasing, i.e., for* $0 < x < y$ *we have* $\log(x) < \log(y)$.

3. *We have the functional equation* $\log(x \cdot y) = \log(x) + \log(y)$ *for all positive real numbers* x *and* y.

Note that the analogue of Theorem 1.1.4. for the logarithm looks more complicated.

One can also derive that the logarithm grows slower than any positive power.

Proposition 1.4. *For any positive real number a we have that*

$$\lim_{x\to 0} \frac{\log(x)}{x^a} = 0.$$

Here is a plot of the logarithm function.

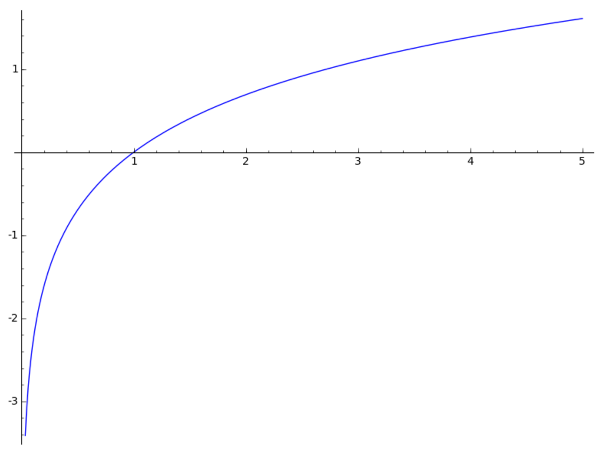

Figure 1.2: The logarithm

1.2 Trigonometric functions

The basic trigonometric functions sine and cosine are usually introduced geometrically, see for example Figure 1.3 (note that we always measure angles in radiants, not in degrees, so a full angle is 2π instead of $360°$). We assume now that the two basic trigonometric functions, $\sin(x)$ and $\cos(x)$, are known. Then we can define the following derived trigonometric functions (wherever

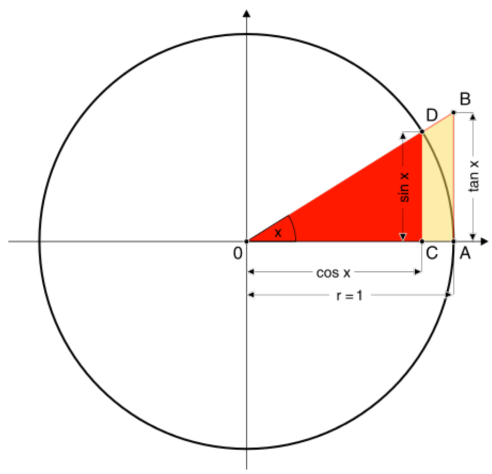

Figure 1.3: The geometric definition of sine, cosine, and tangent

the denominators don't vanish),

$$
\begin{aligned}
\tan(x) &:= \frac{\sin(x)}{\cos(x)}, && \text{the } \textit{tangent} \\
\cot(x) &:= \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)} && \text{the } \textit{cotangent}, \\
\sec(x) &:= \frac{1}{\cos(x)} && \text{the } \textit{secant}, \\
\csc(x) &:= \frac{1}{\sin(x)} && \text{the } \textit{cosecant}.
\end{aligned}
$$

It is clear from the geometric definition that each time one runs around the unit circle in Figure 1.3, the sine and cosine functions behave in the same way and that they are essentially shifts of each other. More precisely, we have the following.

Theorem 1.5. *1. For all $x \in \mathbb{R}$ we have*

$$\sin(x + 2\pi) = \sin(x) \quad \textit{and} \quad \cos(x + 2\pi) = \cos(x).$$

2. For all $x \in \mathbb{R}$ we have

$$\sin(x + \pi) = -\sin(x) \quad \textit{and} \quad \cos(x + \pi) = -\cos(x).$$

3. *For all* $x \in \mathbb{R}$ *we have*

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x).$$

In addition, the sine and cosine functions have certain symmetries.

Proposition 1.6. *The sine function is point symmetric with respect to* 0*, i.e., for all* $x \in \mathbb{R}$ *we have* $\sin(-x) = -\sin(x)$*. The cosine function is axially symmetric, i.e., for all* $x \in \mathbb{R}$ *we have* $\cos(-x) = \cos(x)$*.*

There are a lot of interesting and important identities among the trigonometric functions, some of which we collect in the following theorem.

Theorem 1.7. *Let* x *and* y *be real numbers. Then the following are all true.*

1. $\sin^2(x) + \cos^2(x) = 1$*, where we denote* $\sin^2(x) := (\sin(x))^2$*.*
2. $\sin(x + y) = \cos(x)\sin(y) + \cos(y)\sin(x)$*.*
3. $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$*.*

It is handy to also recall the following special cases of the last two points.

Corollary 1.8. *For all real numbers* x *the following are true.*

1. $\sin(2x) = 2\cos(x)\sin(x)$*,*
2. $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$*.*

Apart from these algebraic relations, the trigonometric functions are also related via differentiation.

Theorem 1.9. *For all* $x \in \mathbb{R}$ *we have* $\sin'(x) = \cos(x)$ *and* $\cos'(x) = -\sin(x)$*.*

It is sometimes important to know some special values of the trigonometric functions. We give a few in the following table.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

From this and the previously mentioned properties of the sine and cosine functions we can now find the zeros and poles of the trigonometric functions introduced here.

Corollary 1.10. *1. We have* $\sin(x) = 0$ *if and only if* $x = \pi k$ *for some integer* $k \in \mathbb{Z}$.

2. We have $\cos(x) = 0$ *if and only if* $x = \pi\left(k + \frac{1}{2}\right)$ *for some integer* $k \in \mathbb{Z}$.

3. The tangent function has (simple) zeros resp. poles exactly in the points of the form $k\pi$ *resp.* $\pi\left(k + \frac{1}{2}\right)$ *with* $k \in \mathbb{Z}$.

It is easy to deduce statements like this for the cotangent, secant and cosecant function as well.

Another important fact about the sine and cosine functions is that they are bounded.

Proposition 1.11. *For all* $x \in \mathbb{R}$ *we have*

$$|\sin(x)| \leq 1 \quad \textit{and} \quad |\cos(x)| \leq 1.$$

In other words, both functions oscillate *between* $+1$ *and* -1.

Here are pictures with plots of the trigonometric functions we have discussed so far (the vertical lines are artefacts which should be ignored).

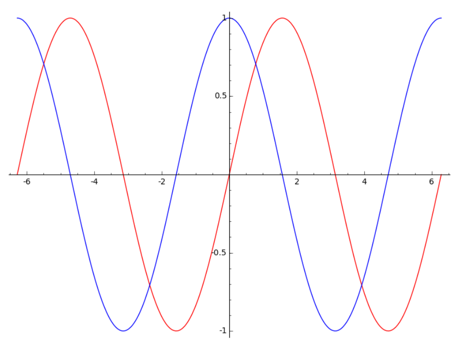

Figure 1.4: The sine (red) and cosine (blue)

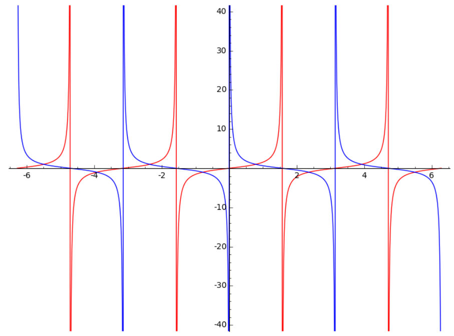

Figure 1.5: The tangent (red) and cotangent (blue)

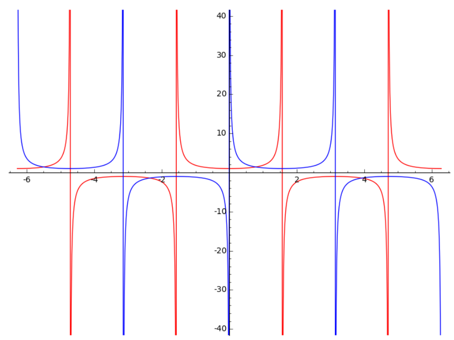

Figure 1.6: The secant (red) and cosecant (blue)

1.3 Hyperbolic functions

The two basic hyperbolic functions are the *hyperbolic sine* and *hyperbolic cosine*, defined by

$$\sinh(x) := \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) := \frac{e^x + e^{-x}}{2}.$$

These also can be motivated geometrically, but we won't go into this here (they take the roles of the sine and cosine in so-called *hyperbolic geometry*, hence the name). These functions are neither periodic nor can they be expressed in terms of each other by shifting the argument[1], so there is not

[1]Note that this is only true for real numbers as arguments

really an analogue for Theorem 1.5. But they have similar symmetries and relations as the usual sine and cosine functions.

Proposition 1.12. *The hyperbolic sine is point symmetric with respect to* 0*, i.e., for all* $x \in \mathbb{R}$ *we have* $\sinh(-x) = -\sinh(x)$*. The hyperbolic cosine is axially symmetric, i.e., for all* $x \in \mathbb{R}$ *we have* $\cosh(-x) = \cosh(x)$*.*

Theorem 1.13. *Let* x *and* y *be real numbers. Then the following are all true.*

1. $\cosh^2(x) - \sinh^2(x) = 1$.
2. $\sinh(x + y) = \cosh(x)\sinh(y) + \cosh(y)\sinh(x)$.
3. $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$.

Corollary 1.14. *For all real numbers* x *the following are true.*

1. $\sinh(2x) = 2\cosh(x)\sinh(x)$,
2. $\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 1 + 2\sinh^2(x)$.

Theorem 1.15. *For all* $x \in \mathbb{R}$ *we have* $\sinh'(x) = \cosh(x)$ *and* $\cosh'(x) = \sinh(x)$*.*

For the hyperbolic functions, there are not really any interesting special values, but we can say the following.

Proposition 1.16.

1. *We have* $\sinh(x) = 0$ *if and only if* $x = 0$*.*
2. *We have* $\cosh(x) \geq 1$ *for all* $x \in \mathbb{R}$*.*

We conclude this section with a plot of the hyperbolic sine and cosine functions.

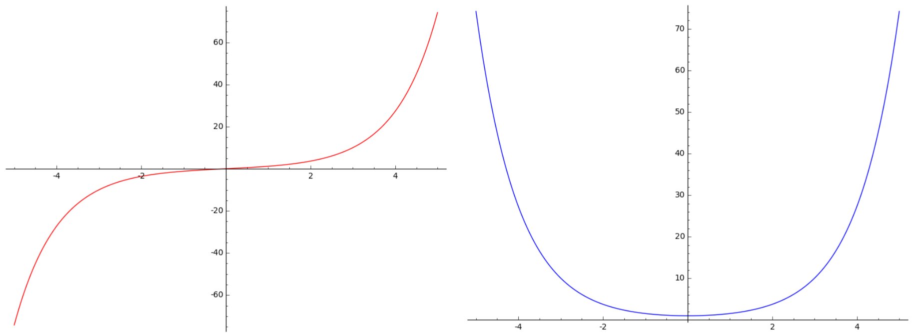

Figure 1.7: Hyperbolic sine (red) and cosine (blue)

2 Systems of linear equations

The problem of solving systems of linear equations is ubiquitous in practically all mathematical disciplines. We start by discussing systems with two equations and two unknowns first, and then we discuss a general method to solve systems of linear equations with arbitrarily many unknowns (focussing on the case of three unknowns).

2.1 Two unknowns

We shall discuss a simple method to solve a system of two linear equations with two unknowns using an example. Say we want to find x and y such that the two equations

$$\begin{array}{ccccc} 2x & -3y & = & 2 \\ x & +y & = & -1 \end{array}$$

are simultaneaously satisfied. One method to deal with this is to isolate one of the unknowns in one of the equations. For example we can rewrite the second equation as

$$y = -x - 1.$$

Now we can plug this into the first equation, thus obtaining one linear equation with only one unknown, which we can easily solve,

$$2x - 3(-x-1) = 2 \quad \Leftrightarrow \quad 5x + 3 = 2 \quad \Leftrightarrow \quad x = -\frac{1}{5}.$$

Thus we know that the only x which solves the system is $x = -\frac{1}{5}$. We plug this back into the rewritten second equation for y, yielding $y = -\frac{4}{5}$. In general, the method works as follows.

Method 2.1.

1. Isolate one variable in one of the two equations (it doesn't matter which equation or which variable one chooses).

2. Plug the expression thus obtained into the other equation, which then contains only one unknown and can be solved directly. This gives the value for the variable which was not isolated in step 1.

3. Plug the obtained values back into equation with one isolated variable to obtain the value for this equation.

In fact, one can give a not too complicated formula for the solution of such a system. Let us say, our unknowns are x and y and the system we want to solve is

$$\begin{array}{rrcl} ax & +by & = & v \\ cx & +dy & = & w, \end{array}$$

where a, b, c, d, v, w are given numbers. In the above example, we would have $a = 2,\ b = -3,\ c = 1,\ d = 1,\ v = 2,\ w = -1$. The general solution formula is then

$$x = \frac{dv - bw}{ad - bc} \quad \text{and} \quad y = \frac{-cv + aw}{ad - bc},$$

provided that the expression $ad - bc$ is not zero (in which case the two equations would be constant multiples of each other).

2.2 Arbitrarily many unknowns

One can easily imagine that as soon as there are more than two unknowns in the game, the above methods become rather cumbersome and one should therefore use a more systematic method then. The basic philosophy is that the fewer unknowns there are in an equation the easier it is to solve. We can do the following three basic things to a system of equation without changing the solution,

1. Interchange two equations,

2. Multiply one equation by a number $a \neq 0$,
3. Add a multiple of one equation to another.

Of course, arbitrary combinations of these operations are also allowed. Now we want to use these operations to successively eliminate unknowns from our equation. Let us look at the following example. Find x, y, z such that the following equations are all satisfied.

$$\begin{array}{rrrcr} x & -y & +z & = & 0 \\ -2x & +y & -3z & = & -7 \\ x & +2y & -z & = & 1. \end{array}$$

We see that if we add 2 times the first equation to the second one, the resulting equation won't contain an x anymore, which is what we want to achieve. This gives the new system

$$\begin{array}{rrrcr} x & -y & +z & = & 0 \\ & -y & -z & = & -7 \\ x & +2y & -z & = & 1. \end{array}$$

Note that we did not change the first equation. Similarly, if we add -1 times the first equation to the third one, the result won't contain an x either,

$$\begin{array}{rrrcr} x & -y & +z & = & 0 \\ & -y & -z & = & -7 \\ & 3y & -2z & = & 1. \end{array}$$

Now the last two equations only contain two variables, so we could solve for y and z with the methods described above, but instead we continue with the elimination process. For convenience, we multiply the second equation by -1 before proceeding,

$$\begin{array}{rrrcr} x & -y & +z & = & 0 \\ & y & +z & = & 7 \\ & 3y & -2z & = & 1. \end{array}$$

If we now add the second equation -3 times to the third, we also eliminate y from that equation,

$$\begin{array}{rrrcr} x & -y & +z & = & 0 \\ & y & +z & = & 7 \\ & & -5z & = & -20. \end{array}$$

Now we divide the third equation by -5, yielding

$$\begin{array}{rrrcr} x & -y & +z & = & 0 \\ & y & +z & = & 7 \\ & & z & = & 4, \end{array}$$

so that we can already read off that $z = 4$. Now our system has *triangular form*, which is what we wanted to achieve. We can now continue the elimination process from below and eliminate the z's from the first and second equation by adding the third equation -1 times to the second and first equation,

$$\begin{array}{rrrcr} x & -y & & = & -4 \\ & y & & = & 3 \\ & & z & = & 4. \end{array}$$

Now we also know that $y = 3$. Now we finish by eliminating y from the first equation by adding the second one to the first one, which gives

$$\begin{array}{rrrcr} x & & & = & -1 \\ & y & & = & 3 \\ & & z & = & 4. \end{array}$$

Now we have got our solution.

This method works in general for systems of linear equations with arbitrarily many variables in the same way.

Method 2.2

1. Assume that the first unknown occurs in the first equation (if not, change the first with one that does contain it).

2. Eliminate the first variable from the all but the first equation by adding multiples of the first equation.

3. Repeat steps 1. and 2. for each equation successively until triangular form is achieved.

4. Starting from the last equation, eliminate the last variable from all but the last equation by adding multiples of the last equation.

5. Repeat step 4. for each equation in reversed order.

6. Read off solutions.

This method is known as the *Gauß elimination method*, named after the famous German mathematician Carl Friedrich Gauß (1777–1855). It can always be used to solve a system of linear equations with arbitrarily many unknowns (or to show that there is no solution) systematically, which makes it a very valuable tool. In the lecture Linear Algebra, this topic is covered in much more depth than here.

2.3 Exercises

Solve the following systems of linear equations.[2]

(a)

$$\begin{array}{rrcr} 2x & -4y & = & -8 \\ 3x & +y & = & 2, \end{array}$$

(b)

$$\begin{array}{rrrcr} x & -3y & +2z & = & -15 \\ -2x & & -4z & = & 0 \\ 3x & +2y & -z & = & 3, \end{array}$$

(c)

$$\begin{array}{rrrcr} & -4y & -z & = & 1 \\ 3x & -2y & & = & 5 \\ -x & +4y & -2z & = & -11, \end{array}$$

(d)

$$\begin{array}{rrrrcr} x & & +z & & = & -3 \\ & y & & -t & = & 3 \\ 3x & & -2z & & = & 16 \\ & -2y & & +5t & = & -3. \end{array}$$

Hint: All solutions are integers.

[2] Upon handing in a complete and correct solution of Exercises (b) and (c) by Monday, Feb. 22, 2016, to be put into my mail box (when you enter the Dept. Math/CS, turn right behind the front desk and enter the first (doorless) room on your right) you can receive up to 3 bonus points which count towards your quiz score.

3 Roots of polynomials

3.1 Quadratic and biquadratic equations

Even though it is probably the most well-known topic among those discussed in these notes, we begin by recalling how to solve *quadratic equations*. Suppose we have an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, c are given real numbers (with $a \neq 0$, otherwise we would in fact be talking about linear equations) and we want to solve for x. Since $a \neq 0$, we can divide the whole equation by it and add a clever zero on the left hand side, giving

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0.$$

The first three summands can easily be recognized to equal $(x + \frac{b}{2a})^2$. This procedure of adding this particular zero is called *completing the square*. Now we reorder the equation to obtain

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Now there are three cases to distinguish,

1. If $\Delta := b^2 - 4ac > 0$, then we obtain two distinct real solutions by taking the square-root, namely

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{\Delta}}{2a}. \tag{3.1}$$

   Note that the square-root of a positive real number a is also positive by definition and therefore unique, while the equation $x^2 = a$ has two solutions, $\sqrt{a}$ and $-\sqrt{a}$.

2. If $\Delta = 0$, then there is precisely one zero,

$$x = -\frac{b}{2a}.$$

   In this case we speak of a *double zero*, since the derivative of the function $f(x) = ax^2 + bx + c$ would also vanish in this case. The zeros in the first case are called *simple zeros*.

3. If $\Delta < 0$, then there is no real solution, since the square of a real number is always non-negative.

Because the behaviour of the quadratic equation is determined entirely by the quantity $\Delta = b^2 - 4ac$, it is called the *discriminant* of the equation (from Latin *discriminare* – to distinguish).

In the case where $a = 1$ (we say that the polynomial is *monic* in this case) and b and c are integers, there is an easy alternative to the above formula, which gives the solutions quicker if they are integers (and in many examples, they will be). It is based on the following observation which goes back to the French-Italian mathematician François Viète (1540–1603).

Lemma 3.1. *Let α_1 and α_2 be roots of the polynomial $x^2 + bx + c$. Then we have $b = -(\alpha_1 + \alpha_2)$ and $c = \alpha_1\alpha_2$.*

Proof. If α_1 and α_2 are the two zeros of our polynomial, then we must have

$$x^2 + bx + c = (x - \alpha_1)(x - \alpha_2) = x^2 - (\alpha_1 + \alpha_2) + \alpha_1\alpha_2.$$

Thus a comparison of the coefficients gives the lemma. □

So if one can factor the number c and combine the divisors so that they sum to $-b$, one also has found the solutions to the equation. This may be easier to do without a calculator than taking square-roots, especially if c has very few divisors. If this factoring method doesn't work, then we also know that our solutions will not be integers.

Sometimes it happens that one has to deal with so-called *biquadratic equations*. Those have the general form

$$ax^4 + bx^2 + c = 0.$$

It is not very complicated to solve these as well, one just substitutes $z = x^2$ to obtain a quadratic equation in z, which one can solve by either one of the above methods. Afterwards, we take the positive and negative square-root of the solutions which are non-negative (the others don't yield any real solutions to the biquadratic equation).

Example 3.2. *Let's solve the biquadratic equation*

$$x^4 + x^2 - 20 = 0.$$

Substituting $z = x^2$ *gives us the quadratic equation*

$$z^2 + z - 20 = 0.$$

The discriminant of this quadratic equation is $\Delta = 1^2 - 4 \cdot 1 \cdot (-20) = 81 > 0$*, therefore, we have two real solutions for* z*, according to our formula* (3.1)*, namely*

$$z = \frac{-1 + \sqrt{81}}{2} = 4 \quad \textit{or} \quad z = \frac{-1 - \sqrt{81}}{2} = -5.$$

Since $z = x^2$*, it must be non-negative if* x *is a real number, so the solution* $z = -5$ *is irrelevant for us and we obtain the two real solutions*

$$x = 2 \quad \textit{or} \quad x = -2.$$

3.1.1 Basics on complex numbers

Let us go back to the third case about solving quadratic equations, when the discriminant of a quadratic equation is negative. The argument why there is no solution is that the square of a real number is always non-negative. But often it is necessary to have a solution to such an equation, even if it is not real. So one *imagines* that there is a "number" which we call i with the property $i^2 = -1$. This number i is **not** a real number, but it is indeed called *imaginary*. We state now that essentially all the rules of arithmetic one is used to from working with real numbers can also be used for this number i. With this, we can in fact solve our quadratic equation

$$ax^2 + bx + c = 0$$

even in the case when the discriminant Δ is negative, namely by writing $\Delta = (-1) \cdot (-\Delta)$, keeping in mind that $-\Delta$ is positive, we can write our formula (3.1) as

$$x = \frac{-b + i\sqrt{-\Delta}}{2a} \quad \text{or} \quad x = \frac{-b - i\sqrt{-\Delta}}{2a}, \tag{3.2}$$

an expression we can now make sense of. In general we call an object of the form $\alpha = a + bi$ with real numbers a and b a *complex number*. The number a is called the *real part* of α, the number b is called the *imaginary part* of α (in particular, the imaginary part of a complex number is always a real number). Every complex number can be simplified to be of this form. We collect a few facts about complex numbers here. We note that one can basically calculate with complex numbers exactly as with real numbers, but since it is not really relevant in the course of the lecture, we won't go into this here. The only thing that we will need is the *exponential* of a complex number. We will just give it as a definition, although it is possible to derive it properly.

Definition 3.3. *For a complex number $\alpha = a + bi$ with real numbers a, b we have*

$$\exp(\alpha) := e^\alpha := e^a \cos(b) + ie^a \sin(b).$$

3.2 Cubic equations

3.2.1 Polynomial division

There is an easy way to divide polynomials by one another which works basically like the long division algorithm for integers. This comes in handy if one has guessed one zero, say α, of a polynomial because one can then divide the polynomial by $x - \alpha$ to obtain a polynomial of lower degree one has to deal with (see Lemma 3.4). We want to divide $x^3 - x^2 - 3x - 9$ by $x - 3$. One only looks at the leading terms of the polynomials and divides those. In this case, we obtain x^2.

$$x^3 \;\; - x^2 - 3x - 9 = \big(x - 3\big)\left(x^2 \qquad\qquad \right).$$

Then we multiply the divisor $x - 3$ by this result and subtract it from the dividend $x^3 - x^2 - 3x - 9$,

$$\begin{array}{l} \;\; x^3 \;\; - x^2 - 3x - 9 = \big(x - 3\big)\left(x^2 \qquad\qquad \right). \\ \underline{-\, x^3 + 3x^2} \end{array}$$

We copy the next lower term downstairs and repeat the procedure with the difference.

$$\begin{array}{l} \;\; x^3 \;\; - x^2 - 3x - 9 = \big(x - 3\big)\left(x^2 \qquad\qquad \right). \\ \underline{-\, x^3 + 3x^2} \\ \qquad\quad 2x^2 - 3x \end{array}$$

Again, we only divide the leading terms, and we get $+2x$, which we write next to the x^2 from the previous step,

$$\begin{array}{l} \;\; x^3 \;\; - x^2 - 3x - 9 = \big(x - 3\big)\left(x^2 + 2x \qquad \right). \\ \underline{-\, x^3 + 3x^2} \\ \qquad\quad 2x^2 - 3x \end{array}$$

We multiply this $2x$ again by our divisor and subtract the result from the dividend,

$$\begin{array}{rl} x^3 \quad - x^2 - 3x - 9 = \left(x-3\right)\left(x^2+2x \qquad\right). \\ -\, x^3 + 3x^2 \qquad\qquad\quad \\ \hline 2x^2 - 3x \\ -\, 2x^2 + 6x \\ \hline \end{array}$$

We repeat this procedure again and finsh up,

$$\begin{array}{l} x^3 \quad - x^2 - 3x - 9 = \left(x-3\right)\left(x^2+2x+3\right). \\ -\, x^3 + 3x^2 \\ \hline \qquad 2x^2 - 3x \\ \qquad -\, 2x^2 + 6x \\ \hline \qquad\qquad 3x - 9 \\ \qquad\qquad -\, 3x + 9 \\ \hline \qquad\qquad\qquad 0 \end{array}$$

Hence we have found our result of the division, namely $x^2 + 2x + 3$.

In general, it is not necessary that the polynomial division goes through without a remainder. If there is one, it will have a degree less than the divisor. We want to divide $x^4 - 3x^3 + 2x^2 - 5x + 7$ by $x^2 - x + 1$. Up to the last step everything is the same as before,

$$\begin{array}{l} x^4 - 3x^3 + 2x^2 - 5x + 7 = \left(x^2 - x + 1\right)\left(x^2 - 2x - 1\right) \qquad\qquad . \\ -\, x^4 \;\; + x^3 \;\; - x^2 \\ \hline \qquad -\, 2x^3 \;\; + x^2 - 5x \\ \qquad\quad 2x^3 - 2x^2 + 2x \\ \hline \qquad\qquad\quad -\, x^2 - 3x + 7 \\ \qquad\qquad\qquad x^2 \;\; - x + 1 \\ \hline \qquad\qquad\qquad\quad -\, 4x + 8 \end{array}$$

We see that the last difference is a polynomial of degree 1, but not 0. This polynomial is our remainder, which we have to add on the right-hand side,

$$x^4 - 3x^3 + 2x^2 - 5x + 7 = \left(x^2 - x + 1\right)\left(x^2 - 2x - 1\right) - 4x + 8.$$

$$\begin{array}{l}
\underline{-x^4 + x^3 - x^2} \\
\quad -2x^3 + x^2 - 5x \\
\quad \underline{2x^3 - 2x^2 + 2x} \\
\qquad -x^2 - 3x + 7 \\
\qquad \underline{x^2 - x + 1} \\
\qquad\quad -4x + 8
\end{array}$$

3.2.2 Cubic and higher degree polynomials

As indicated in the last section, we can use polynomial division to find roots of polynomials. This is based on two easy facts which we recall in the following lemma.

Lemma 3.4. 1. *The product of two numbers is* 0 *if and only one of the numbers is zero,*

$$a \cdot b = 0 \quad \Rightarrow \quad a = 0 \quad \textit{or} \quad b = 0.$$

2. *A number α is the root of a polynomial $p(x)$ if and only if $x - \alpha$ divides $p(x)$, i.e. the polynomial division of $p(x)$ by $x - \alpha$ goes through without a remainder.*

So if we want to find the roots or zeros of a cubic polynomial, we can somehow guess one zero, and use polynomial division to obtain a quadratic polynomial, whose zeros we can determine using the formulas in Section 3.1. The question now is how we can guess a zero of a polynomial. At least in case of a monic polynomial with integer coefficients, there is a very easy observation with which we can check at least for integer zeros. It is a generalization of the Theorem of Vieta (Lemma 3.1)

Proposition 3.5. *If an integer a is a zero of a monic polynomial with integer coefficients, then it divides the absolute term of the polynomial. If no divisor of the absolute term is a zero of the polynomial, then all zeros are irrational.*

So if we cannot find an integer root, then we have essentially no chance of "guessing" one. In this case, there are numerical methods to obtain approximations for zeros, the most well-known goes back to Sir Isaac Newton

(1642–1727), or (for polynomials of degree 3 and 4) there are even closed formulas. None of these will be relevant in our course.

Example 3.6. *We want to determine all zeros of the cubic polynomial*

$$x^3 + 4x^2 - 7x - 10.$$

By Proposition 3.5, we have to check the divisors of the absolute term, which is 10 *in this case. By trial and error we find that that* 2 *is in fact a zero of this polynomial. Now we use polynomial division,*

$$\begin{array}{rrrl} x^3 + 4x^2 & - 7x & - 10 & = \big(x-2\big)\big(x^2+6x+5\big). \\ \underline{-\,x^3 + 2x^2} & & & \\ 6x^2 & - 7x & & \\ \underline{-\,6x^2} & \underline{+\,12x} & & \\ & 5x & - 10 & \\ & \underline{-\,5x} & \underline{+\,10} & \\ & & 0 & \end{array}$$

The result is $x^2 + 6x + 5$. The zeros of this polynomial are -1 and -5 which one can check either using (3.1) *or Lemma 3.1. Thus we have the three zeros $x = 2$, $x = -1$, and $x = -5$.*

3.3 Exercises

Carry out polynomial division for the following pairs of polynomials.

1. $x^2 - 4x + 3$ divided by $x - 3$,
2. $x^3 - 4x^2 + 3x - 12$ divided by $x - 4$,
3. $x^3 - 5x^2 + 3x - 4$ divided by $x^2 + 1$,
4. $x^4 + 7x^2 - 4x + 2$ divided by $x^2 + 3x - 1$.

Find all zeros (real and complex) of the following polynomials.[3]

1. $x^2 - 4x + 4$,
2. $x^2 - 4x + 13$,
3. $x^4 - 25x^2 + 144$,
4. $x^3 - 2x^2 - 5x + 6$,
5. $x^3 - 2x^2 + 9x - 18$.

[3]Upon handing in a complete and correct solution of Exercises (c) and (e) by Monday, Feb. 22, 2016, to be put into my mail box (when you enter the Dept. Math/CS, turn right behind the front desk and enter the first (doorless) room on your right) you can receive up to 3 bonus points which count towards your quiz score.

4 Differentiation

4.1 Basic rules

Here we recall the basic rules of differentiation without going into the motivation and definition of the derivative of a function. We refer to the relevant calculus textbooks for this. We denote the derivative of a function f with respect to the variable x by

$$\frac{d}{dx}f(x) = f'(x).$$

Theorem 4.1. *Let f and g be differentiable functions on an open interval to the real numbers, and α, β be real numbers. Then the following are true.*

1. *The derivative in* linear*, i.e.*

$$(\alpha f(x) + \beta g(x))' = \alpha f'(x) + \beta g(x).$$

2. *The derivative obeys the* Leibniz rule*,*

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

3. *We have the* chain rule*,*

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

4. *Wherever $g(x) \neq 0$, we have the* quotient rule*,*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

An important property of real functions that we are interested in is monotonicity, for example because a strictly monotonic function is invertible. The derivative is a good tool to test for this.

Theorem 4.2. *Let f be a differentiable function on an open interval I. If $f'(x) \geq 0$ (resp. $f(x) \leq 0$) for all $x \in I$ then f is monotonically increasing (resp. decreasing). If the inequality is always a strict one, we also have strict monotonicity, in particular, there is a function f^{-1} such that $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$ for all $x \in I$ and all y in the image of f.*

Note that f^{-1} does **not** indicate the reciprocal, but the *inverse* of f. We can also say something about the derivative of that inverse.

Proposition 4.3. *Let f be an invertible, differentiable function with $f'(x) > 0$ and inverse f^{-1}. Then f^{-1} is differentiable as well and we have*

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}.$$

Proof. We know that $f(f^{-1}(x)) = x$, so if we differentiate this equation, applying the chain rule, we get

$$f'(f^{-1}(x)) \cdot (f^{-1}(x))' = 1.$$

Rearranging this equation yields the claim. $\square$

Example 4.4. *For $-\frac{\pi}{2} < x < \frac{\pi}{2}$ we have, according to the quotient rule, that*

$$\tan'(x) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = 1 + \tan^2(x),$$

which is obviously positive. Thus the tangent function is invertible on this interval, the inverse is called the arcus tangent*, denoted by* $\arctan(x)$*. By Proposition 4.3, its derivative is therefore given by*

$$\arctan'(x) = \frac{1}{1 + \tan^2(\arctan(x))} = \frac{1}{1 + x^2}.$$

This may at first glance be rather unexpected, for the tangent is a transcendental, complicated function, and yet, the derivative of its inverse is a rational function.

4.2 Interpretations

In various applications it is important to know interpretations for the derivative. Generally speaking, the derivative always describes a rate of change. In the following table, we give a small dictionary how one can interpret the derivative of a function depending on what the function describes (in most applications, the variable stands for time, which is why we denote it by t in this section).

Interpretation for $f(t)$	Interpretation for $f'(t)$
abstract function	slope of the graph at every point
position of a particle	velocity of a particle
velocity of a particle	acceleration of a particle
momentum of a particle	force acting on a particle
electric charge	electric current

4.3 Derivatives of some special functions

Here we give a table of the derivatives of some of the most important elementary functions. The letter x shall always denote the variable, and a a fixed real number (if not specified otherwise).

$f(x)$	$f'(x)$
x^a	ax^{a-1}
e^x	e^x
a^x, $a > 0$	$\ln(a)a^x$
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$

4.4 Exercises

Find the first derivatives of the following functions.[4]

(a) $x^{17} - 4x^8 + 5x^3 - 7x + 5$,

(b) $\sin(x^2)$,

(c) $e^{2x}\cos(3x)$,

(d) e^{x^2},

(e) $\ln(\arctan(2^x))$.

Use Proposition 4.3 to derive the formulas for the derivatives of the functions

(a) $\ln(x)$,

(b) $\arcsin(x)$,

(c) $\arccos(x)$.

[4] Upon handing in a complete and correct solution of Exercises (b), (c), and (e) by Monday, Feb. 22, 2016, to be put into my mail box (when you enter the Dept. Math/CS, turn right behind the front desk and enter the first (doorless) room on you right) you can receive up to 3 bonus points which count towards your quiz score.

5 Integration

5.1 The Principal Theorem of Calculus

Geometrically, the integral of a (let's say continuous) function over an interval is the sum of the oriented areas under the graph of the function. Already from this notion, one can derive several important properties of the integral.

Proposition 5.1. *Let f, g be continuous functions on a closed interval $[a, b]$*[5] *and α, β be real numbers.*

1. *We have*

$$\int_a^b (\alpha f(x) + \beta g(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx,$$

   i.e., the integral is linear.

2. *For any $c \in [a, b]$ we have*

$$\int_c^c f(x)dx = 0.$$

3. *We have*

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

4. *For $c \in [a, b]$ we have*

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$$

At first glance it is rather surprising that this is related to derivatives. The Principal Theorem of Calculus makes this precise.

Theorem 5.2. *Let f be a continuous function on a closed interval $[a, b]$.*

1. *The function $F(x) := \int_a^x f(t)dt$ for $x \in [a, b]$ is an antiderivative of f, i.e. $F'(x) = f(x)$.*

[5] By an interval $[a, b]$ for real numbers $a \leq b$ we mean the set of all real numbers x such that $a \leq x \leq b$.

2. *Let F be any antiderivative of f, i.e. a differentiable function with $F'(x) = f(x)$. Then we have*

$$\int_a^b f(x)dx = [F(x)]_a^b := F(b) - F(a).$$

At this point, we want to stress that there is more than one antiderivative to a continuous function.

Lemma 5.3. *Let F and G be two antiderivatives of a continuous function f. Then their difference $F(x) - G(x)$ is constant.*

5.2 Basic rules and techniques

With the Principal Theorem of Calculus and the differentiation rules in Theorem 4.1 we can derive several rules for integration as well.

Theorem 5.4. *Let f and g be differentiable functions on an interval $[a, b]$. Then we have the following rules of integration,*

1. *the rule of* partial integration*,*

$$\int_a^b f'(x)g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx,$$

2. *the* first substitution rule*,*

$$\int_a^b f(g(x))g'(x)dx = [F(g(x))]_a^b,$$

where F is an antiderivative of f.

3. *if g is invertible, then we have the* second substitution rule*,*

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))g'(t)dt.$$

4. *A special case of the substitution is the* logarithmic integration*:*

$$\int_a^b \frac{f'(x)}{f(x)}dx = [\ln|f(x)|]_a^b.$$

We give an example for the application of these rules.

Example 5.5. 1. *We use partial integration to find an antiderivative of the logarithm function. By Theorem 5.2, one is given by*

$$\int_1^x \ln(t)dt = \int_1^x 1 \cdot \ln(t)dt.$$

With partial integration it is important to make a good choice, which of the functions one chooses for $f'(x)$ and which one for $g(x)$. Theoretically, this choice doesn't matter but for computational purposes, there is often one good and one bad choice. In general it is a good call to choose the function for $f'(x)$ of which we know an antiderivative. So here we choose $f'(x) = 1$ (and we can therefore use $f(x) = x$) and $g(x) = \ln(x)$ (and therefore $g'(x) = \frac{1}{x}$). Thus we have

$$\int_1^x 1 \cdot \ln(t)dt = [t\ln(t)]_1^x - \int_1^x t \cdot \frac{1}{t}dt = x\ln(x) - [t]_1^x = x\ln(x) - x + 1.$$

Since additive constants do not matter in this context (cf. Lemma 5.3), we can pick the function

$$F(x) = x\ln(x) - x$$

as an antiderivative.

2. *Using the first substitution rule, we want to find an antiderivative of the function $f(x) = xe^{x^2}$. Despite the fact that this is the product of two functions, partial integration will not yield a closed expression for the antiderivative. The reason for this is essentially that an antiderivative of e^{x^2} – even though it certainly exists – is not expressible in terms of so-called elementary functions. However, substitution will give us an answer in this case. By Theorem 5.2, an antiderivative is given by*

$$\int_0^x te^{t^2}dt = \frac{1}{2}\int_0^x (2t)e^{t^2}dt = \frac{1}{2}[e^{t^2}]_0^x = \frac{1}{2}e^{x^2} - \frac{1}{2}.$$

Again, we get rid of the additive constant, yielding the antiderivative

$$F(x) = \frac{1}{2}e^{x^2}.$$

3. *The function* $f(x) = \sqrt{1-x^2}$ *is defined for* $-1 \le x \le 1$ *and its graph describes the upper half of the unit circle. Thus we know from elementary geometry that*

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \frac{\pi}{2}.$$

We want to derive this with the methods of calculus, more precisely, with the second substitution rule. We substitute $x = \sin(t)$ *and use the second substitution rule. We obtain*

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \int_{\arcsin(-1)}^{\arcsin(1)} \sqrt{1-\sin^2(t)}\cos(t)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(t)dt.$$

To compute this integral, we use partial integration to obtain an antiderivative of $\cos^2(t)$. *We have*

$$\begin{aligned}\int_0^t \cos^2(s)ds &= [\cos(s)\sin(s)]_0^t + \int_0^t \sin^2(t)dt \\ &= \cos(t)\sin(t) + \int_0^t (1-\cos^2(s))ds \\ &= \cos(t)\sin(t) + t - \int_0^t \cos^2(s)ds.\end{aligned}$$

By rearranging this equality we find

$$\int_0^t \cos^2(s)ds = \frac{1}{2}\cos(t)\sin(t) + \frac{t}{2}.$$

Thus by Theorem 5.2 we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(t)dt = \left[\frac{1}{2}\cos(t)\sin(t) + \frac{t}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

Another important technique for integrating rational functions (i.e. quotients of polynomials) is the method of *partial fraction decomposition*. It is also very useful in other contexts. Suppose we want to integrate a function $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and we assume that the degree of p is less than that of $q(x)$. If that assumption is not satisfied we can use polynomial division (see Section 3.2.1) to write as a sum of a polynomial (which is easy to integrate) and a rational function of the assumed form. We can write $q(x)$ as a product of linear and quadratic polynomials and write $\frac{p(x)}{q(x)}$ as a sum of rational functions with these factors as denominators. The point is that one can rather easily integrate functions of this form.

We illustrate this with an example.

Example 5.6. *We want to find an antiderivative for the function*

$$f(x) = \frac{x^2 - 20x + 39}{(x-1)(x+3)(x-4)}.$$

As described above, we want to write this as a function of the form

$$f(x) = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-4}$$

for real numbers A, B, C. In order to find these numbers, we write this all as one fraction again,

$$\begin{aligned} f(x) =& \frac{A(x+3)(x-4) + B(x-1)(x-4) + C(x-1)(x+3)}{(x-1)(x+3)(x-4)} \\ =& \frac{(A+B+C)x^2 + (-A-5B+2C)x + (-12A+4B-3C)}{(x-1)(x+3)(x-4)}. \end{aligned}$$

By comparing coefficients we see that A, B, C have to satisfy the following system of linear equations,

$$\begin{array}{cccccc} A & +B & +C & = & 1 \\ -A & -5B & +2C & = & -20 \\ -12A & +4B & -3C & = & 39. \end{array}$$

Using the method discussed in Section 2.2, we can solve this and find that $A = -2$, $B = 4$, and $C = -1$, wherefore we have

$$f(x) = -\frac{2}{x-1} + \frac{4}{x+3} - \frac{1}{x-4}.$$

It is now easy to find an antiderivative, for example the function

$$F(x) = -2\ln|x-1| + 4\ln|x+3| - \ln|x-4|$$

will work.

5.3 Indefinite and improper integrals

So far, we have only considered definite integrals, i.e. integrals with specific limits where the integrand function is continuous on the full closed interval over which we integrate. There is also a notion of so-called *indefinite* integrals which don't have specified limits. For a continuous function f we mean by the expression

$$\int f(x)dx$$

the collection of **all** antiderivatives of f. So, strictly speaking, this expression is a set of functions, not a single function. In particular, there is no formally correct way to plug in an argument into this. If we want to work with a specific antiderivative of f, we need to work with a definite integral, where the upper limit is considered a variable, as in Theorem 5.2. Sometimes, one is a bit sloppy though and uses the symbol $\int f(x)dx$ for the "most straightforward" antiderivative of f, i.e. one where there is no visible additive constant. For example, one would write $\int \cos(x)dx = \sin(x)$ instead of, e.g. $\sin(x) + 17$. But one should always be aware of the proper meaning of the improper integral.

Apart from this, one also sometimes needs to consider integrals, where one (or both) of the limits are either ∞ or $-\infty$, or also singularities of the function one is integrating (e.g. 0 with the function $\frac{1}{\sqrt{x}}$). The definition of this is as follows

Definition 5.7. *Let f be a continuous function on the half-open interval $[a, b)$, the set of all real numbers x with $a \leq x < b$, where b is allowed to be $+\infty$. Then we define the* improper integral *of $f(x)$ over the interval $[a, b]$ by*

$$\int_a^b f(x)dx := \lim_{c \nearrow b} \int_a^c f(x)dx$$

and say that it exists if the limit exists. Analogously, if f is continouous on the half-open interval $(a, b]$ where a is allowed to be $-\infty$, then we define the improper integral

$$\int_a^b f(x)dx := \lim_{c \searrow a} \int_c^b f(x)dx.$$

Note that in the above definition the expression $\int_a^c f(x)dx$ is a proper, definite integral as we have discussed before, while the expression $\int_a^b f(x)dx$ does not even need to make sense.

It is a very delicate question in general whether an improper integral exists. As a rule of thumb one can say (at least for functions without sign changes) that the improper integral exists if f decays fast enough if the problematic limit is $\pm\infty$ or goes to infinity slowly enough if the problematic limit is finite.

Proposition 5.8. *Let a be a positive real number.*

1. *The improper integral*
$$\int_1^\infty x^{-a}dx$$
exists if and only if $a > 1$. Its value is $\frac{1}{a-1}$.

2. *The improper integral*
$$\int_0^1 x^{-a}dx$$
exists if and only if $0 < a < 1$. Its value is $\frac{1}{1-a}$.

3. *The improper integral*
$$\int_0^\infty e^{-x}dx$$
exists and has the value 1.

These integrals can be computed directly and then it is easy to take the limit. But in the vast majority of cases, this is not possible. One important criterion to decide existence of improper integrals (but not to compute them) is given in the following theorem.

Theorem 5.9. *Let f be a continuous function on the half-open interval $[a, b)$, where b is allowed to be $+\infty$. If there exists a continuous function g on $[a, b)$ such that $g(x) \geq |f(x)|$ for all $x \in [c, b)$ for some $c \in [a, b)$ and the improper integral $\int_a^b g(x)dx$ exists, then so does the improper integral $\int_a^b f(x)dx$.*

Example 5.10. *We want to show that the improper integral*
$$\int_0^\infty e^{-x^2}dx$$
exists. As pointed out before, we cannot explicitly compute an antiderivative in order to calculate the limit, so we need to do something else. In fact we

know that for $x \geq 1$ we have $e^{-x^2} \leq e^{-x}$ since the exponential function is monotonically increasing and that

$$\int_0^\infty e^{-x} dx = 1$$

exists. Thus we know by Theorem 5.9 that also the improper integral $\int_0^\infty e^{-x^2} dx$ exists. Note that this procedure does not give any clue about the actual value of this integral (which is $\frac{\sqrt{\pi}}{2} \approx 0.8862269$).

5.4 Exercises

Compute the following integrals.

(a) $\int_0^4 x^2 - x - 3dx$,

(b) $\int_0^\pi \sin(x)dx$,

(c) $\int_0^2 \pi \cos(x)$,

(d) $\int_{-3}^3 \sinh(x)dx$.

Find an antiderivative to each of the following functions.[6]

(a) $(\ln(x))^2$,

(b) $\sin^2(x)$,

(c) $\cos(x)e^x$,

(d) $\frac{x^2-6x+12}{x^3-x^2+4x-17}$,

(e) $\tan(x)$,

(f) $\cosh(x^4 - x^2)(2x^3 - x)$,

(g) $\cos(2x - 3)$,

(h) $\sqrt{1 + x^2}$.

[6]Upon handing in a complete and correct solution of Exercises (b), (c), and (f) by Monday, Feb. 22, 2016, to be put into my mail box (when you enter the Dept. Math/CS, turn right behind the front desk and enter the first (doorless) room on you right) you can receive up to 3 bonus points which count towards your quiz score.