Special values of shifted convolution Dirichlet series

Michael H. Mertens
(joint work with Ken Ono and Kathrin Bringmann)

Emory University

29th Automorphic Forms Workshop,
Ann Arbor, March 05, 2015

1. Introduction

2. Nuts and bolts
   - Harmonic Maaß forms
   - Rankin-Cohen brackets
   - Poincaré series

3. Holomorphic projection

4. The results and examples

Table of Contents

Shifted convolution Dirichlet series

Definitions

- Let $f_1 \in S_{k_1}(\Gamma_0(N))$ and $f_2 \in S_{k_2}(\Gamma_0(N))$ $(k_1 \geq k_2)$ with

$$f_i(\tau) = \sum_{n=1}^{\infty} a_i(n) q^n.$$

Shifted convolution Dirichlet series

Definitions

- Let $f_1 \in S_{k_1}(\Gamma_0(N))$ and $f_2 \in S_{k_2}(\Gamma_0(N))$ ($k_1 \geq k_2$) with

$$f_i(\tau) = \sum_{n=1}^{\infty} a_i(n) q^n.$$

- Rankin-Selberg convolution

$$L(f_1 \otimes f_2, s) := \sum_{n=1}^{\infty} \frac{a_1(n)\overline{a_2(n)}}{n^s},$$

Shifted convolution Dirichlet series

Definitions

- Let $f_1 \in S_{k_1}(\Gamma_0(N))$ and $f_2 \in S_{k_2}(\Gamma_0(N))$ $(k_1 \geq k_2)$ with

$$f_i(\tau) = \sum_{n=1}^{\infty} a_i(n) q^n.$$

- Rankin-Selberg convolution

$$L(f_1 \otimes f_2, s) := \sum_{n=1}^{\infty} \frac{a_1(n)\overline{a_2(n)}}{n^s},$$

- shifted convolution Dirichlet series (Hoffstein-Hulse, 2013)

$$D(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)\overline{a_2(n)}}{n^s}.$$

Shifted convolution Dirichlet series

Definitions (continued)

- derived shifted convolution series

$$D^{(\mu)}(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)\overline{a_2(n)}(n+h)^{\mu}0}{n^s}.$$

Shifted convolution Dirichlet series

Definitions (continued)

- derived shifted convolution series

$$D^{(\mu)}(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)\overline{a_2(n)}(n+h)^{\mu}0}{n^s}.$$

- use to define symmetrized shifted convolution Dirichlet series $\widehat{D}^{(\nu)}(f_1, f_2, h; s)$, e.g. for $\nu = 0$ and $k_1 = k_2$,

$$\widehat{D}^{(0)}(f_1, f_2, h; s) = D(f_1, f_2, h; s) - D(\overline{f_2}, \overline{f_1}, -h; s),$$

Shifted convolution Dirichlet series

Definitions (continued)

- derived shifted convolution series

$$D^{(\mu)}(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)\overline{a_2(n)}(n+h)^{\mu}0}{n^s}.$$

- use to define symmetrized shifted convolution Dirichlet series $\widehat{D}^{(\nu)}(f_1, f_2, h; s)$, e.g. for $\nu = 0$ and $k_1 = k_2$,

$$\widehat{D}^{(0)}(f_1, f_2, h; s) = D(f_1, f_2, h; s) - D(\overline{f_2}, \overline{f_1}, -h; s),$$

- generating function of special values

$$\mathbb{L}^{(\nu)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \widehat{D}^{(\nu)}(f_1, f_2, h; k_1 - 1) q^h.$$

A numerical conundrum

$$\mathbb{L}^{(0)}(\Delta,\Delta;\tau)$$
$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots$$

A numerical conundrum

$$\mathbb{L}^{(0)}(\Delta,\Delta;\tau)$$
$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots$$

- define real numbers $\alpha = 106.10455\ldots$ and $\beta = 2.8402\ldots$, and the weight 12 weakly holomorphic modular form

$$\sum_{n=-1}^{\infty} r(n)q^n := -\Delta(\tau)(j(\tau)^2 - 1464j(\tau) - \alpha^2 + 1464\alpha),$$

A numerical conundrum

$$\mathbb{L}^{(0)}(\Delta,\Delta;\tau)$$
$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots$$

- define real numbers $\alpha = 106.10455\ldots$ and $\beta = 2.8402\ldots$, and the weight 12 weakly holomorphic modular form

$$\sum_{n=-1}^{\infty} r(n)q^n := -\Delta(\tau)(j(\tau)^2 - 1464j(\tau) - \alpha^2 + 1464\alpha),$$

- play around a bit and find

$$-\frac{\Delta}{\beta}\left(\frac{65520}{691} + \frac{E_2}{\Delta} - \sum_{n\neq 0} r(n)n^{-11}q^n\right)$$
$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots$$

Table of Contents

Harmonic Maaß forms

Definition

Let $f : \mathbb{H} \to \mathbb{C}$ be a real-analytic function and $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$ and $N \in \mathbb{N}$ with

1. $f|_{2-k}\gamma = f$ for all $\gamma \in \Gamma_0(N)$,
2. $\Delta_{2-k} f \equiv 0$ with $\mathbb{H} \ni \tau = x + iy$ and

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

3. f grows at most linearly exponentially at the cusps of $\Gamma_0(N)$.

Then f is called a harmonic Maaß form of weight $2-k$ for $\Gamma_0(N)$. The $\mathbb{C}$-vector space of of these forms is denoted by $H_{2-k}(\Gamma_0(N))$.

Harmonic Maaß forms

Lemma

For $f \in H_{2-k}(\Gamma_0(N))$ we have the splitting

$$f(\tau) = \sum_{m=m_0}^{\infty} c_f^+(n)q^n + \frac{(4\pi y)^{1-k}}{k-1}\overline{c_f^-(0)} + \sum_{\substack{n=n_0\\n\neq 0}}^{\infty} \overline{c_f^-(n)} n^{k-1}\Gamma(1-k;4\pi ny)q^{-n}.$$

Harmonic Maaß forms

Proposition (Bruinier-Funke)

$$\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to M_k^!(\Gamma_0(N)),\ f \mapsto 2iy^{2-k}\overline{\frac{\partial f}{\partial \overline{\tau}}}$$

is well-defined and surjective with kernel $M_{2-k}(\Gamma_0(N))$. Moreover, we have

$$(\xi_{2-k}f)(\tau) = -(4\pi)^{k-1}\sum_{n=n_0}^{\infty} c_f^-(n)q^n.$$

Harmonic Maaß forms

Proposition (Bruinier-Funke)

$$\xi_{2-k}: H_{2-k}(\Gamma_0(N)) \to M_k^!(\Gamma_0(N)),\ f \mapsto 2iy^{2-k}\overline{\frac{\partial f}{\partial \overline{\tau}}}$$

is well-defined and surjective with kernel $M_{2-k}(\Gamma_0(N))$. Moreover, we have

$$(\xi_{2-k}f)(\tau) = -(4\pi)^{k-1}\sum_{n=n_0}^{\infty} \overline{c_f^-(n)}q^n.$$

- $-(4\pi)^{1-k}\xi_{2-k}f$ is called the shadow of f.

Harmonic Maaß forms

Proposition (Bruinier-Funke)

$$\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to M_k^!(\Gamma_0(N)),\ f \mapsto 2iy^{2-k}\overline{\frac{\partial f}{\partial \overline{\tau}}}$$

is well-defined and surjective with kernel $M_{2-k}(\Gamma_0(N))$. Moreover, we have

$$(\xi_{2-k}f)(\tau) = -(4\pi)^{k-1}\sum_{n=n_0}^{\infty} c_f^-(n)q^n.$$

- $-(4\pi)^{1-k}\xi_{2-k}f$ is called the shadow of f.
- for $f_1 \in S_{k_1}(\Gamma_0(N))$ denote by M_{f_1} a HMF with shadow f_1

Rankin-Cohen brackets

Definition

Let $f, g : \mathbb{H} \to \mathbb{C}$ be smooth functions on the upper half-plane and $k, \ell \in \mathbb{R}$ be some real numbers, the weights of f and g. Then for a non-negative integer ν we define the νth Rankin-Cohen bracket of f and g by

$$[f,g]_\nu := \frac{1}{(2\pi i)^\nu} \sum_{\mu=0}^{\nu} (-1)^\mu \binom{k+\nu-1}{\nu-\mu} \binom{\ell+\nu-1}{\mu} \frac{\partial^\mu f}{\partial \tau^\mu} \frac{\partial^{\nu-\mu} g}{\partial \tau}.$$

Rankin-Cohen brackets

Definition

Let $f, g : \mathbb{H} \to \mathbb{C}$ be smooth functions on the upper half-plane and $k, \ell \in \mathbb{R}$ be some real numbers, the weights of f and g. Then for a non-negative integer ν we define the νth Rankin-Cohen bracket of f and g by

$$[f,g]_\nu := \frac{1}{(2\pi i)^\nu} \sum_{\mu=0}^{\nu} (-1)^\mu \binom{k+\nu-1}{\nu-\mu} \binom{\ell+\nu-1}{\mu} \frac{\partial^\mu f}{\partial \tau^\mu} \frac{\partial^{\nu-\mu} g}{\partial \tau}.$$

- f, g modular of weights $k, \ell \Rightarrow [f,g]_\nu$ modular of weight $k+\ell+2\nu$.

Poincaré series

- A general Poincaré series of weight k for $\Gamma_0(N)$:

$$\mathbb{P}(m, k, N, \varphi_m; \tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi_m^* |_k \gamma)(\tau),$$

where $\varphi_m^*(\tau) := \varphi_m(y) \exp(2\pi i m x)$.

Poincaré series

- A general Poincaré series of weight k for $\Gamma_0(N)$:

$$\mathbb{P}(m,k,N,\varphi_m;\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi_m^* |_k \gamma)(\tau),$$

  where $\varphi_m^*(\tau) := \varphi_m(y)\exp(2\pi i m x)$.

- two special cases ($m > 0$):

$$P(m,k,N;\tau) := \mathbb{P}(m,k,N,e^{-my};\tau) \in S_k(\Gamma_0(N)),$$
$$Q(-m,k,N;\tau) := \mathbb{P}(-m,2-k,N,\mathcal{M}_{1-\frac{k}{2}}(-4\pi m y);\tau) \in H_{2-k}(\Gamma_0(N)$$

  where $\mathcal{M}$ is defined in terms of the M-Whittaker function.

Relations among Poincaré series

Lemma

If $k \geq 2$ is even and $m, N \geq 1$, then

$$\xi_{2-k}(Q(-m,k,N;\tau)) = (4\pi)^{k-1}m^{k-1}(k-1)\cdot P(m,k,N;\tau) \in S_k(\Gamma_0(N)).$$

Table of Contents

Idea of holomorphic projection

- $\Phi : \mathbb{H} \to \mathbb{C}$ continuous, transforming like a modular form of weight $k \geq 2$ for some $\Gamma_0(N)$, moderate growth at cusps (Attention for $k = 2!$).

Idea of holomorphic projection

- $\Phi : \mathbb{H} \to \mathbb{C}$ continuous, transforming like a modular form of weight $k \geq 2$ for some $\Gamma_0(N)$, moderate growth at cusps (Attention for $k = 2!$).
- The map $f \mapsto \langle f, \Phi \rangle$ defines a linear functional on $S_k(\Gamma_0(N))$.

Idea of holomorphic projection

- $\Phi : \mathbb{H} \to \mathbb{C}$ continuous, transforming like a modular form of weight $k \geq 2$ for some $\Gamma_0(N)$, moderate growth at cusps (Attention for $k = 2!$).
- The map $f \mapsto \langle f, \Phi \rangle$ defines a linear functional on $S_k(\Gamma_0(N))$.
- $\Rightarrow \quad \exists! \tilde{\Phi} \in S_k(\Gamma_0(N))$ s.t. $\langle \cdot, \Phi \rangle = \langle \cdot, \tilde{\Phi} \rangle$

Idea of holomorphic projection

- $\Phi : \mathbb{H} \to \mathbb{C}$ continuous, transforming like a modular form of weight $k \geq 2$ for some $\Gamma_0(N)$, moderate growth at cusps (Attention for $k = 2!$).
- The map $f \mapsto \langle f, \Phi \rangle$ defines a linear functional on $S_k(\Gamma_0(N))$.
- $\Rightarrow \quad \exists! \tilde{\Phi} \in S_k(\Gamma_0(N))$ s.t. $\langle \cdot, \Phi \rangle = \langle \cdot, \tilde{\Phi} \rangle$
- This $\tilde{\Phi}$ is (essentially) the holomorphic projection of Φ.

Idea of holomorphic projection

- $\Phi : \mathbb{H} \to \mathbb{C}$ continuous, transforming like a modular form of weight $k \geq 2$ for some $\Gamma_0(N)$, moderate growth at cusps (Attention for $k = 2!$).
- The map $f \mapsto \langle f, \Phi \rangle$ defines a linear functional on $S_k(\Gamma_0(N))$.
- $\Rightarrow \quad \exists! \tilde{\Phi} \in S_k(\Gamma_0(N))$ s.t. $\langle \cdot, \Phi \rangle = \langle \cdot, \tilde{\Phi} \rangle$
- This $\tilde{\Phi}$ is (essentially) the holomorphic projection of Φ.
- same reasoning works for regularized Petersson inner product $\rightsquigarrow$ regularized holomorphic projection.

Fourier coefficients

Definition

If $\Phi(\tau) = \sum_{n\in\mathbb{Z}} a_\Phi(n,y)q^n$, $(y = \mathrm{Im}(\tau))$, then

$(\pi_{hol}\Phi)(\tau) := (\pi_{hol}^{(k)}\Phi)(\tau) := \sum_{n=0}^{\infty} c(n)q^n$, where

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!}\int_0^\infty a_\Phi(n,y)e^{-4\pi n y}y^{k-2}dy, \qquad n>0.$$

Properties of holomorphic projection

Proposition

- If Φ is holomorphic, then $\pi_{hol}\Phi = \Phi$.

Properties of holomorphic projection

Proposition

- If Φ is holomorphic, then $\pi_{hol}\Phi = \Phi$.
- If Φ transforms like a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$, on some group $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, then $\pi_{hol}\Phi \in M_k(\Gamma)$.

Properties of holomorphic projection

Proposition

- If Φ is holomorphic, then $\pi_{hol}\Phi = \Phi$.
- If Φ transforms like a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$, on some group $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, then $\pi_{hol}\Phi \in M_k(\Gamma)$.
- The operator π_{hol} commutes with all the operators $U(N)$, $V(N)$, and $S_{N,r}$ (sieving operator).

Properties of holomorphic projection

Proposition

- If Φ is holomorphic, then $\pi_{hol}\Phi = \Phi$.
- If Φ transforms like a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$, on some group $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, then $\pi_{hol}\Phi \in M_k(\Gamma)$.
- The operator π_{hol} commutes with all the operators $U(N)$, $V(N)$, and $S_{N,r}$ (sieving operator).

Remark

- For $k = 2$, $\pi_{hol}\Phi$ is a quasi-modular form of weight 2.

Properties of holomorphic projection

Proposition

- If Φ is holomorphic, then $\pi_{hol}\Phi = \Phi$.
- If Φ transforms like a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$, $k > 2$, on some group $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, then $\pi_{hol}\Phi \in M_k(\Gamma)$.
- The operator π_{hol} commutes with all the operators $U(N)$, $V(N)$, and $S_{N,r}$ (sieving operator).

Remark

- For $k = 2$, $\pi_{hol}\Phi$ is a quasi-modular form of weight 2.
- For the regularized holomorphic projection, weakly holomorphic forms are possible images

Holomorphic projection of mixed mock modular forms

Let

$$G_{a,b}(X,Y) := \sum_{j=0}^{a-2} (-1)^j \binom{a+b-3}{a-2-j} \binom{j+b-2}{j} X^{a-2-j} Y^j \in \mathbb{C}[X,Y]$$

Holomorphic projection of mixed mock modular forms

Let

$$G_{a,b}(X,Y) := \sum_{j=0}^{a-2} (-1)^j \binom{a+b-3}{a-2-j} \binom{j+b-2}{j} X^{a-2-j} Y^j \in \mathbb{C}[X,Y]$$

Proposition (Zagier)

Let $f_1 \in S_{k_1}(\Gamma_0(N))$ and $f_2 \in S_{k_2}(\Gamma_0(N))$ be cusp forms as before. Then we have for $0 \leq \nu \leq \frac{k_1-k_2}{2}$ that

$$\pi_{hol}^{reg}([M_{f_1}, f_2]_\nu)(\tau) = [M_{f_1}^+, f_2]_\nu(\tau) - (k_1-2)! \sum_{h=1}^{\infty} q^h \left[\sum_{n=1}^{\infty} a_2(n+h)\overline{a_1(n)} \right.$$

$$\times \sum_{\mu=0}^{\nu} \binom{\nu-k_1+1}{\nu-\mu} \binom{\nu+k_2-1}{\mu} \Big((n+h)^{-\nu-k_2+1} G_{2\nu-k_1+k_2+2, k_1-\mu}(n+h, n)$$

$$\left. - n^{\mu-k_1+1}(n+h)^{\nu-\mu} \Big) \right].$$

Table of Contents

The theorems

Theorem 1 (M.-Ono)

If $0 \leq \nu \leq \frac{k_1 - k_2}{2}$, then

$$\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1-2)!} \cdot [M_{f_1}^{+}, f_2]_\nu + F,$$

where $F \in \widetilde{M}^!_{2\nu+2-k_1+k_2}(\Gamma_0(N))$. Moreover, if M_{f_1} is good for f_2, then $F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$.

The theorems

Theorem 1 (M.-Ono)

If $0 \leq \nu \leq \frac{k_1-k_2}{2}$, then

$$\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1-2)!} \cdot [M_{f_1}^+, f_2]_\nu + F,$$

where $F \in \widetilde{M}^!_{2\nu+2-k_1+k_2}(\Gamma_0(N))$. Moreover, if M_{f_1} is good for f_2, then $F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$.

- M_{f_1} is good for f_2 if $[M_{f_1}^+, f_2]_\nu$ grows at most polynomially at all cusps (very rare phenomenon).

The theorems

Theorem 1 (M.-Ono)

If $0 \leq \nu \leq \frac{k_1 - k_2}{2}$, then

$$\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1-2)!} \cdot [M_{f_1}^+, f_2]_\nu + F,$$

where $F \in \widetilde{M}^!_{2\nu+2-k_1+k_2}(\Gamma_0(N))$. Moreover, if M_{f_1} is good for f_2, then $F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$.

- M_{f_1} is good for f_2 if $[M_{f_1}^+, f_2]_\nu$ grows at most polynomially at all cusps (very rare phenomenon).
- $\widetilde{M}^!_k(\Gamma_0(N))$ is the weakly holomorphic extension of

$$\widetilde{M}_k(\Gamma_0(N)) = \begin{cases} M_k(\Gamma_0(N)) & \text{if } k \geq 4, \\ \mathbb{C}E_2 \oplus M_2(\Gamma_0(N)) & \text{if } k = 2. \end{cases}$$

An example

Let $f_1 = f_2 = \Delta = \frac{1}{\beta} P(1, 12, 1; \tau)$

An example

Let $f_1 = f_2 = \Delta = \frac{1}{\beta} P(1, 12, 1; \tau)$

$$\beta := \frac{(4\pi)^{11}}{10!} \cdot \|P(1, 12, 1)\|^2 = 1 + 2\pi \sum_{c=1}^{\infty} \frac{K(1, 1, c)}{c} \cdot J_{11}(4\pi/c) = 2.8402 \ldots.$$

An example

Let $f_1 = f_2 = \Delta = \frac{1}{\beta} P(1,12,1;\tau)$

$$\beta := \frac{(4\pi)^{11}}{10!} \cdot \|P(1,12,1)\|^2 = 1 + 2\pi \sum_{c=1}^{\infty} \frac{K(1,1,c)}{c} \cdot J_{11}(4\pi/c) = 2.8402\ldots.$$

$Q(-1,12,1;\tau) = Q^+(-1,12,1;\tau) + Q^-(-1,12,1;\tau) \in H_{-10}(\mathrm{SL}_2(\mathbb{Z}))$, the canonical preimage of $P(1,12,1;\tau)$ under ξ_{-10} (up to a constant factor), is good for Δ

An example

Let $f_1 = f_2 = \Delta = \frac{1}{\beta} P(1, 12, 1; \tau)$

$$\beta := \frac{(4\pi)^{11}}{10!} \cdot \|P(1, 12, 1)\|^2 = 1 + 2\pi \sum_{c=1}^{\infty} \frac{K(1, 1, c)}{c} \cdot J_{11}(4\pi/c) = 2.8402 \ldots.$$

$Q(-1, 12, 1; \tau) = Q^+(-1, 12, 1; \tau) + Q^-(-1, 12, 1; \tau) \in H_{-10}(\mathrm{SL}_2(\mathbb{Z}))$, the canonical preimage of $P(1, 12, 1; \tau)$ under ξ_{-10} (up to a constant factor), is good for Δ

$$\mathbb{L}^{(0)}(\Delta, \Delta; \tau) = \frac{Q^+(-1, 12, 1; \tau) \cdot \Delta(\tau)}{11! \cdot \beta} - \frac{E_2(\tau)}{\beta}$$
$$= -33.383 \ldots q + 266.439 \ldots q^2 - 1519.218 \ldots q^3 + 4827.434 \ldots q^4 - \ldots.$$

p-adic properties

Theorem 2 (Bringmann-M.-Ono)

Let $f \in S_k(\Gamma_0(N))$ be an even weight newform. If p is a prime with $p^2 \mid N$, then there exist constants $\delta_1, \delta_2 \in \mathbb{C}$, a weight 2 weakly holomorphic quasimodular form $\mathcal{Q}_f \in \widetilde{M}_2^!(\Gamma_0(N))$, and a weight $2-k$ weakly holomorphic p-adic modular form $\mathcal{L}_f$ for which

$$\mathbb{L}(f, f; \tau) = \delta_1 f(\tau)\mathcal{L}_f(\tau) + \delta_2 f(\tau)\mathcal{E}_f(\tau) + \mathcal{Q}_f(\tau).$$

Moreover, if f has complex multiplication, then there are choices with $\delta_2 = 0$.

p-adic properties

Theorem 2 (Bringmann-M.-Ono)

Let $f \in S_k(\Gamma_0(N))$ be an even weight newform. If p is a prime with $p^2 \mid N$, then there exist constants $\delta_1, \delta_2 \in \mathbb{C}$, a weight 2 weakly holomorphic quasimodular form $\mathcal{Q}_f \in \widetilde{M}_2^!(\Gamma_0(N))$, and a weight $2-k$ weakly holomorphic p-adic modular form $\mathcal{L}_f$ for which

$$\mathbb{L}(f,f;\tau) = \delta_1 f(\tau)\mathcal{L}_f(\tau) + \delta_2 f(\tau)\mathcal{E}_f(\tau) + \mathcal{Q}_f(\tau).$$

Moreover, if f has complex multiplication, then there are choices with $\delta_2 = 0$.

- $\mathcal{E}_F(\tau)$ is the (holomorphic) Eichler integral of $F(\tau) = \sum_{n\in\mathbb{Z}} A(n)q^n$,

$$\mathcal{E}_F(\tau) := \sum_{n\neq 0} A(n)n^{1-k}q^n.$$

p-adic properties

Theorem 3 (Bringmann-M.-Ono)

Let f be as in Theorem 2. Then the following are all true.

1. We have that f may be expressed as a finite linear combination of the form
$$f(\tau) = \sum_{p \nmid m} \alpha_m P(m, k, N; \tau),$$
with $\alpha_m \in \mathbb{C}$.
2. In terms of the linear combination in (1), if $Q(\tau) := \sum_{p \nmid m} \frac{\alpha_m}{m^{k-1}} Q(-m, k, N; \tau)$, then
$$\xi_{2-k}(Q) = (4\pi)^{k-1}(k-1)f.$$
3. If $Q^+(\tau) = \sum_n a_Q(n)q^n$, then $a_Q(pn) = 0$ for all $n \in \mathbb{N}$.
4. We have that $D^{k-1}(Q^+) \in M_k^!(\Gamma_0(N))$.

Another example

- Let $f(\tau) = \eta(3\tau)^8 \in S_4^{new}(\Gamma_0(9))$. f hast CM by $\mathbb{Q}(\sqrt{-3})$.

Another example

- Let $f(\tau) = \eta(3\tau)^8 \in S_4^{new}(\Gamma_0(9))$. f hast CM by $\mathbb{Q}(\sqrt{-3})$.
- Numerics

h	3	6	9	12
$\widehat{D}(f,f,h;3)$	$-10.7466\ldots$	$12.7931\ldots$	$6.4671\ldots$	$-79.2777\ldots$

Another example

- Let $f(\tau) = \eta(3\tau)^8 \in S_4^{new}(\Gamma_0(9))$. f hast CM by $\mathbb{Q}(\sqrt{-3})$.
- Numerics

h	3	6	9	12
$\widehat{D}(f,f,h;3)$	$-10.7466\ldots$	$12.7931\ldots$	$6.4671\ldots$	$-79.2777\ldots$

- Let

$$\beta := \frac{(4\pi)^3}{2} \cdot \|P(1,4,9)\|^2 = 1.0468\ldots, \quad \gamma := -0.0796\ldots,$$

$$\delta := -0.8756\ldots. \qquad N.B.: \frac{\delta}{\gamma} = 11$$

Anoter example (continued)

- We find by Theorem 1 that

$$\mathbb{L}(f,f;\tau) = \frac{f(\tau)Q^+(-1,4,9;\tau)}{\beta}$$

$$+\gamma\left(1-24\sum_{n=1}^{\infty}\sigma_1(3n)q^{3n}\right)+\delta\left(1+12\sum_{n=1}^{\infty}\sum_{\substack{d|3n\\3\nmid d}}dq^{3n}\right).$$

Anoter example (continued)

- We find by Theorem 1 that

$$\mathbb{L}(f,f;\tau) = \frac{f(\tau)Q^+(-1,4,9;\tau)}{\beta} + \underbrace{\sum_{n=0}^{\infty} b_f(n)q^n}_{=:\mathcal{Q}_f(\tau)}$$

- In Theorem 2 we find $\delta_1 = \frac{1}{\beta}$,

$$\mathcal{L}_f(\tau) := Q^+(-1,4,9;\tau) = q^{-1} - \frac{1}{4}q^2 + \frac{49}{125}q^5 - \frac{3}{32}q^8 - \cdots = -\mathcal{E}_m(\tau),$$

with

$$m(\tau) := \left(\frac{\eta(\tau)^3}{\eta(9\tau)^3} + 3\right)^2 \cdot \eta(3\tau)^8 = q^{-1} + 2q^2 - 49q^5 + 48q^8 + \ldots.$$

Another example (continued)

- $f\mathcal{L}_f$ is a cuspidal 3-adic modular form of weight 2, $f\mathcal{L}_f \equiv 1 \pmod 3$ as q-series

Another example (continued)

- $f\mathcal{L}_f$ is a cuspidal 3-adic modular form of weight 2, $f\mathcal{L}_f \equiv 1 \pmod{3}$ as q-series
- This yields

$$\begin{aligned}
\widehat{D}(f,f,h;3) - b_f(h) &\in \frac{3}{\beta}\cdot\mathbb{Z}_{(3)},\\
\widehat{D}(f,f,9h+6;3) - b_f(9h+6) &\in \frac{9}{\beta}\cdot\mathbb{Z}_{(3)},\\
\widehat{D}(f,f,36h+30;3) - b_f(36h+30)) &\in \frac{27}{\beta}\cdot\mathbb{Z}_{(3)},\\
&\cdots
\end{aligned}$$

Another example (continued)

- $f\mathcal{L}_f$ is a cuspidal 3-adic modular form of weight 2, $f\mathcal{L}_f \equiv 1 \pmod 3$ as q-series
- This yields

$$\begin{aligned}
\widehat{D}(f,f,h;3) - b_f(h) &\in \frac{3}{\beta}\cdot \mathbb{Z}_{(3)},\\
\widehat{D}(f,f,9h+6;3) - b_f(9h+6) &\in \frac{9}{\beta}\cdot \mathbb{Z}_{(3)},\\
\widehat{D}(f,f,36h+30;3) - b_f(36h+30)) &\in \frac{27}{\beta}\cdot \mathbb{Z}_{(3)},\\
&\dots
\end{aligned}$$

- the (rational) numbers $\beta(\widehat{D}(f,f,h;3) - b_f(h))$ are 'almost always' multiples of any fixed power of 3.

Thank you for your attention.