Special values of shifted convolution Dirichlet series

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(joint work with Ken Ono)

Universität zu Köln

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- Slides for Jeremy’s talk are available at

  `http://users.wfu.edu/rouseja/2adic/bristol.pdf`.

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Shifted convolution Dirichlet series

Definitions

- $f_1 \in S_{k_1}(\Gamma_0(N))$, $f_2 \in S_{k_2}(\Gamma_0(N))$ with

$$f_i(\tau) = \sum_{n=1}^{\infty} a_i(n) q^n.$$

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- shifted convolution series

$$D(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)\overline{a_2(n)}}{n^s}.$$

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Definitions (continued)

- derived shifted convolution series

$$D^{(\mu)}(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)\overline{a_2(n)}(n+h)^{\mu}}{n^s}.$$

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- use to define symmetrized shifted convolution Dirichlet series $\widehat{D}^{(\nu)}(f_1, f_2, h; s)$, e.g. for $\nu = 0$ and $k_1 = k_2$

$$\widehat{D}^{(0)} = \widehat{D}(f_1, f_2, h; s) = D(f_1, f_2, h; s) - D(\overline{f_2}, \overline{f_1}, -h; s),$$

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- generating function of special values

$$\mathbb{L}^{(\nu)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \widehat{D}^{(\nu)}(f_1, f_2, h; k_1 - 1) q^h$$

A numerical conundrum

$$\mathbb{L}^{(0)}(\Delta,\Delta;\tau)$$
$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots$$

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- define real numbers $\alpha = 106.10455\ldots$, $\beta = 2.8402\ldots$ and the weight 12 modular form

$$-\Delta(j^2 - 1464j - \alpha^2 + 1464\alpha) =: \sum_{n=-1}^{\infty} r(n)q^n$$

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- play around a bit and find

$$-\frac{\Delta}{\beta}\left(\frac{65520}{691} + \frac{E_2}{\Delta} - \sum_{n\neq 0} r(n)n^{-11}q^n\right)$$
$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots$$

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Harmonic Maaß forms

Definition

Let $f : \mathbb{H} \to \mathbb{C}$ be a real-analytic function and $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$ with

1. $f|_{2-k}\gamma = f$ for all $\gamma \in \Gamma_0(N)$
2. $\Delta_{2-k} f \equiv 0$ with $\mathbb{H} \ni \tau = x + iy$ and

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

3. f grows at most linearly exponentially at the cusps of $\Gamma_0(N)$.

Then f is called a harmonic Maaß form (HMF) of weight $2-k$ on $\Gamma_0(N)$, which are the elements of the vector space $H_{2-k}(\Gamma_0(N))$.

Harmonic Maaß forms

Lemma

For $f \in H_{2-k}(\Gamma_0(N))$ we have the splitting

$$\sum_{n=m_0}^{\infty} c_f^+(n) q^n + \frac{(4\pi y)^{1-k}}{k-1} \overline{c_f^-(0)} + \sum_{\substack{n=n_0 \\ n\neq 0}}^{\infty} \overline{c_f^-(n)} n^{k-1} \Gamma(1-k; 4\pi n y) q^{-n}.$$

Harmonic Maaß forms

Proposition (Bruinier-Funke)

$$\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to M_k^!(\Gamma_0(N)),\ f \mapsto \xi_{2-k} f := 2iy^{2-k}\overline{\frac{\partial f}{\partial \overline{\tau}}}$$

is well-defined and surjective with kernel $M_{2-k}^!(\Gamma_0(N))$. Moreover, we have

$$(\xi_{2-k} f)(\tau) = -(4\pi)^{k-1} \sum_{n=n_0}^{\infty} c_f^-(n) q^n.$$

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- $-(4\pi)^{1-k}\xi_{2-k} f$: shadow of f
- for $f_1 \in S_{k_1}$ denote by M_{f_1} a HMF with shadow f_1

Rankin-Cohen brackets

Definition

Let $f, g : \mathbb{H} \to \mathbb{C}$ be smooth functions on the upper half-plane and $k, \ell \in \mathbb{R}$ be some real numbers, the weights of f and g. Then for a non-negative integer ν we define the νth Rankin-Cohen bracket of f and g by

$$[f,g]_\nu := \frac{1}{(2\pi i)^\nu} \sum_{\mu=0}^{\nu} (-1)^\mu \binom{k+\nu-1}{\nu-\mu} \binom{\ell+\nu-1}{\mu} \frac{\partial^\mu f}{\partial \tau^\mu} \frac{\partial^{\nu-\mu} g}{\partial \tau}.$$

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- f, g modular of weights $k, \ell \;\Rightarrow\; [f,g]_\nu$ modular of weight $k+\ell+2\nu$.

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- $\tilde{f} : \mathbb{H} \to \mathbb{C}$ smooth (possibly non-holomorphic) modular form of weight $k \geq 2$ on $\Gamma_0(N)$ (with moderate growth at cusps).

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- explicit formula for the Fourier coefficients of f in terms of those of $\tilde{f}$
- same reasoning for regularized Petersson inner product also works, growth conditions can be weakened

Holomorphic projection of mixed mock modular forms

Let

$$G_{a,b}(X,Y) := \sum_{j=0}^{a-2} (-1)^j \binom{a+b-3}{a-2-j} \binom{j+b-2}{j} X^{a-2-j} Y^j \in \mathbb{C}[X,Y].$$

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Proposition (Zagier)

Let $f_1 \in S_{k_1}(\Gamma_0(N))$ and $f_2 \in S_{k_2}(\Gamma_0(N))$ be cusp forms of even weights as in the introduction and let $M_{f_1} \in H_{2-k_1}(\Gamma_0(N))$ be a harmonic Maass form with shadow f_1. then we have

$$\pi_{hol}^{reg}([M_{f_1}, f_2]_\nu)(\tau) = [M_{f_1}^+, f_2]_\nu(\tau)$$

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$$-(k_1-2)! \sum_{\mu=0}^{\nu} \binom{\nu-k_1+1}{\nu-\mu} \binom{\nu+k_2-1}{\mu} \sum_{h=1}^{\infty} q^h \left[\sum_{n=1}^{\infty} a_2(n+h)\overline{a_1(n)} \right.$$

$$\left. \times \left((n+h)^{-\nu-k_2+1} G_{2\nu-k_1+k_2+2,k_1-\mu}(n+h,n) - n^{\mu-k_1+1}(n+h)^{\nu-\mu} \right) \right].$$

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The theorem

Theorem (M.-Ono)

If $0 \leq \nu \leq \frac{k_1-k_2}{2}$, then

$$\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1-2)!} \cdot [M_{f_1}^{+}, f_2]_\nu + F,$$

where $F \in \widetilde{M}^{!}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$. Moreover, if M_{f_1} is good for f_2, then $F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$.

- M_{f_1} is good for f_2, if $[M_{f_1}, f_2]_\nu$ grows at most polynomially at the cusps (very rare phenomenon)
- $\widetilde{M}^{!}_{k}(\Gamma_0(N))$ is the weakly holomorphic extension of

$$\widetilde{M}_k(\Gamma_0(N)) = \begin{cases} M_k(\Gamma_0(N)) & \text{if } k \geq 4, \\ \mathbb{C}E_2 \oplus M_2(\Gamma_0(N)) & \text{if } k = 2. \end{cases}$$

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$$\mathbb{L}^{(0)}(\Delta, \Delta; \tau) = \frac{Q^+(-1, 12, 1; \tau) \cdot \Delta(\tau)}{11! \cdot \beta} - \frac{E_2(\tau)}{\beta}$$
$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots$$

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$\rightsquigarrow$ efficient way to compute $\widehat{D}(\Delta, \Delta, h; 11)$

Example II

Let $f = f_1 = f_2 = \eta(3\tau)^8 = \frac{1}{\beta} P(1, 4, 9; \tau) \in S_4(\Gamma_0(9))$. f has CM by $\mathbb{Q}(\sqrt{-3})$

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h	3	6	9	12
$\widehat{D}(f,f,h;3)$	$-10.7466\ldots$	$12.7931\ldots$	$6.4671\ldots$	$-79.2777\ldots$

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Let

$$\beta := \frac{(4\pi)^3}{2}\cdot\|P(1,4,9)\|^2 = 1.0468\ldots, \quad \gamma = -0.0796\ldots, \quad \delta = -0.8756\ldots$$

and

$$T(f;h) := \beta\widehat{D}(f,f,h;3) + 24\beta\gamma\sum_{d|h} d - 12\beta\delta\sum_{\substack{d|h\\ 3\nmid d}} d.$$

Example II (continued)

h	3	6	9	12
$T(f;h)$	$-8.250\dots$	$22.391\dots$	-8.229	-61.992

Example II (continued)

h	3	6	9	12
$T(f;h)$	$\sim -\frac{33}{4}$	$\sim \frac{2799}{125}$	$\sim -\frac{32919}{4000}$	$\sim -\frac{8250771}{133100}$

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Theorem yields

$$\mathbb{L}^{(0)}(f,f;\tau) - \frac{Q^+(-1,4,9;\tau)f(\tau)}{\beta}$$

$$=\gamma\left(1-24\sum_{n=1}^{\infty}\sigma_1(3n)q^{3n}\right)+\delta\left(1+12\sum_{n=1}^{\infty}\sum_{\substack{d|3n\\ 3\nmid d}} dq^{3n}\right).$$

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we know from work of Bruinier-Ono-Rhoades that

$$Q^+(-1,4,9;\tau) = q^{-1} - \frac{1}{4}q^2 + \frac{49}{125}q^5 - \frac{3}{32}q^8 - \dots$$

has all rational Fourier coefficients.

Thank you for your attention.