Mixed mock modular forms are vector-valued modular forms

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joint work in progress with Martin Raum

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Ramanujan's deathbed letter

S. Ramanujan (1887-1920)

The modern definition

Definition

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Problem

Multiplying holomorphic functions is natural, multiplying harmonic functions is usually a bad idea.

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$M_k^{[d]}(\Gamma)$: Space of modular forms of order d and weight k.

Higher depth modular forms II

- twists of Eisenstein series by modular symbols

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Question

Is there a unified framework for (mixed) mock modular forms and higher order modular forms?

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Symmetric powers

Definition

For $d \geq 0$ define the d^{th} symmetric power representation of $\mathrm{SL}_2(\mathbb{R})$ denoted by sym^d by

$$\mathrm{sym}^d(g)p(X) := p(X)|_{-d}\, g^{-1} = (-cX+a)^d p\left(\frac{dX-b}{-cX+a}\right),$$

where $p(X) \in \mathbb{C}[X]$, $\deg p \leq d$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.

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Fact

Complex, irreducible, finite-dimensional representations of $\mathrm{SL}_2(\mathbb{R})$ are exhausted by sym^d.

Universal parabolic extensions

Notation

For two arithmetic types ρ, ρ' and an extension class $\varphi \in \mathrm{Ext}^1(\rho, \rho')$ let $\rho \boxplus_\varphi \rho'$ denote the extension corresponding to φ, i.e. we have the short exact sequence

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$$\rho \hookrightarrow \rho \boxplus_{\mathrm{pb}}^d \rho' \twoheadrightarrow \rho' \otimes \mathrm{sym}^d \otimes \mathrm{Ext}_{\mathrm{pb}}^1(\rho, \rho' \otimes \mathrm{sym}^d)$$

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$$\rho \hookrightarrow \rho \boxplus^d_{\mathrm{pb}} \rho' \twoheadrightarrow \rho' \otimes \mathrm{sym}^d \otimes \mathrm{Ext}^1_{\mathrm{pb}}(\rho, \rho' \otimes \mathrm{sym}^d)$$

- For each $\varphi \in \mathrm{Ext}^1_{\mathrm{pb}}(\rho, \rho' \otimes \mathrm{sym}^d) \setminus \{0\}$ we have a direct summand

$$\rho \boxplus_\varphi \rho' \otimes \mathrm{sym}^d \leq \rho \boxplus^d_{\mathrm{pb}} \rho'.$$

Socle series

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  - $\mathrm{soc}^0(\rho) = \{0\}$,
  - $\mathrm{soc}^j(\rho) = \mathrm{soc}(\rho / \mathrm{soc}^{j-1}(\rho))$ for all $1 \leq j \leq d$.

Virtually real-arithmetic types

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We call an arithmetic type ρ real-arithmetic if its socle factors are direct sums of symmetric powers. We call ρ virtually real-arithmetic (vra) type if its restriction to a finite index subgroup is real-arithmetic.

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- If ρ, ρ' have finite index kernel, then $\rho \boxplus^{d}_{\mathrm{pb}} \rho'$ is a vra type.

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Classical modular forms I

Proposition 1

Let ρ be an arithmetic type and $k \in \mathbb{Z}$. Then the map

$$R_k : \mathscr{C}_k^\infty(\rho) \to \mathscr{C}_{k+1}^\infty(\mathrm{std} \otimes \rho),\ f \mapsto (X - \tau)\partial_\tau f - kf$$

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- $\mathscr{C}_k^\infty(\rho)$: smooth functions with $|_{k,\rho}$ action
- $\operatorname{std} = \operatorname{sym}^1$: standard representation of $\mathrm{SL}_2(\mathbb{R})$, $V(\operatorname{std}) \cong \mathbb{C}[X]_{\leq 1}$.

Classical modular forms II

Theorem 1 (Kuga-Shimura, 1960; M.-Raum, 2017)

For $\kappa, d \in \mathbb{N}_0$, $k \in \mathbb{Z}$ and an arithmetic type ρ define the map

$$p_\kappa R_k^d : \mathscr{C}_k^\infty(\rho) \to \mathscr{C}_{k+d-\kappa}^\infty(\mathrm{sym}^{d+\kappa} \otimes \rho),\ f \mapsto (X - \tau)^\kappa R_k^d f.$$

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If $[\mathrm{SL}_2(\mathbb{Z}) : \mathrm{Kern}(\rho)] < \infty$, $k+d$ is odd and $k > d$ or $k < -d$, then

$$\bigoplus_{\substack{j=-d \\ j \equiv d\,(2)}}^{d} M_{k+j}(\rho) \to M_k(\mathrm{sym}^d \otimes \rho), \quad (f_{-d}, \ldots, f_d) \mapsto \sum_{\substack{j=-d \\ j \equiv d\,(2)}}^{d} p_{\frac{d+j}{2}} R_{k+j}^{\frac{d-j}{2}} f_j$$

is an isomorphism.

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Cocycles

For $f \in \mathscr{C}_k^\infty(\rho)$, $k \in \mathbb{Z}$, we define its cocycle by

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Fact (Fay, Bruinier-Funke,...)

Let ρ be an arithmetic type with finite index kernel.

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- For $f \in \mathbb{M}_\ell(\rho) \otimes M_k^!(\rho')$, we have

$$\varphi_f = f|_{k+\ell,\rho\otimes\rho'}(1-\gamma) \in \mathbb{C}[\tau]_{\leq -\ell} \otimes V(\rho) \otimes M_k^!(\rho'), \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

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Theorem 2 (M. - Raum)

Let $d \in 2\mathbb{N}_0$ and ρ an arithmetic type with finite index kernel. There is a well-defined map

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If $k \in \mathbb{Z}$ and ρ' with finite index kernel, let

$$\mathcal{I}^!_k(\rho') = \sum f(\tau) f^\vee \in M^!_k(\rho') \otimes M^!_k(\rho')^\vee,$$

where f runs through a basis of $M^!_k(\rho')$ and $f^\vee$ is the dual of f. Then there is a map

$$\begin{aligned} \mathbb{M}_{-d}(\rho) \otimes M^!_k(\rho') &\to M^!_{k-d}(\rho\rho' \boxplus^d_{\mathrm{pb}} \rho' M^!_k(\rho')^\vee), \\ f &\mapsto f \boxplus (X-\tau)^d \mathcal{I}^!_k(\rho') \otimes \varphi_f. \end{aligned}$$

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Higher order modular forms

Let $\mathbb{1}^{[0]} := \mathbb{1}$ and define for $d > 0$ the representation $\mathbb{1}^{[d]}$ of $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ recursively by

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Theorem 3 (M.-Raum)

Let $d \geq 0$ and $k \in \mathbb{Z}$. Then the map

$$M_k(\mathbb{1}^{[d]}) \to M_k^{[d]}, \; f \boxplus * \mapsto f$$

is surjective. In particular, we have

$$M_k(\mathbb{1}^{[d]}) \cong \bigoplus_{j=0}^{d} M_k^{[j]} \otimes \mathrm{H}(\Gamma, \mathbb{1})^{\otimes(d-j)}.$$

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Thank you for your attention.