Automorphism Groups of Hyperbolic Lattices

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- *Automorphism group*:

$$\mathrm{Aut}(L) = \{g \in \mathrm{GL}(\mathcal{V}) \mid Lg = L \text{ and } \Phi(xg, yg) = \Phi(x, y) \text{ for all } x, y \in L\}$$

$$\mathrm{Aut}_{\mathbb{Z}}(A) = \{g \in \mathrm{GL}_n(\mathbb{Z}) \mid gAg^{tr} = A\}$$

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For a given Gram matrix A of a lattice (L, Φ) determine a generating system for $\mathrm{Aut}_{\mathbb{Z}}(A)$.

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- E.B. Vinberg (1972): algorithm for reflective hyperbolic lattices (those lattices are very rare).
- R.E. Borcherds, J.H. Conway ...($\sim$1988): results for special lattices, e.g. $(\mathbb{Z}^n, diag(-1, 1, ..., 1))$ for small n.

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(DC.1) $\sigma(x, y) > 0$ for all $x \in \mathcal{V}_1^{>0}$, $y \in \mathcal{V}_2^{>0}$.

(DC.2) For any $x \in \mathcal{V} \setminus \mathcal{V}_1^{>0}$ there is a $y \in \mathcal{V}_2^{\geq 0} \setminus \{0\}$ such that $\sigma(x, y) \leq 0$. ($\mathcal{V}_i^{\geq 0} := \overline{\mathcal{V}_i^{>0}}$).

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(vi) $P_D = \{x \in \mathcal{V}_1^{>0} \mid x \, D\text{-perfect}, \, \mu_D(x) = 1\}$

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- *Facet* $\mathcal{F}(r)$ von $\mathcal{D}_D(x)$:

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Proposition

$x \in P_D$, $r \notin \mathcal{V}_1^{\geq 0}$ direction of x.

$$\Rightarrow \quad y = x + \rho r \in P_D \text{ for a } \rho > 0.$$

$x + \rho r$ is the *neighbour* of x in direction r. x, y are called *contiguous*.

D-Voronoi Graph

Γ_D graph with

$$V(\Gamma_D) = P_D$$
$$E(\Gamma_D) = \{(x, y) \in P_D^2 \mid x, y \text{ contiguous}\}$$

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$$\begin{aligned} V(\Gamma_D) =& P_D \\ E(\Gamma_D) =& \{(x,y) \in P_D^2 \mid x,y \text{ contiguous}\} \end{aligned}$$

- Γ_D is connected, i.e. the D-Voronoi-domains cover $\mathcal{V}_2^{>0}$ (exact face-to-face tesselation).

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- Γ_D is locally finite, i.e. every D-perfect point has only finitely many neighbours.

Lemma

$\Omega \leq \mathrm{GL}(\mathcal{V})$ discrete subgroup acting properly discontinuously on $\mathcal{V}_1^{>0}$, $D\Omega^{ad} \subseteq D$. Then Ω acts on Γ_D as well.

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Theorem (Opgenorth, 2001)

Let $D \subset \mathcal{V}_2^{\geq 0} \setminus \{0\}$ admissible and discrete in $\mathcal{V}_2$ and $\Omega \leq \mathrm{GL}(\mathcal{V})$ as above and let the residue class graph Γ_D/Ω be finite. Further let $X := \{x_1, \ldots, x_\ell\}$ a transversal of D-perfect points, which span a subtree T of Γ_D and let T_1 denote the neighbourhood of T in Γ_D. For all $y \in T_1$ choose an $\omega_y \in \Omega$, such that $\omega_y^{-1}(y) \in X$. Then

$$\Omega = \langle \omega_y, \mathrm{Stab}_\Omega\, x_i \mid y \in T_1, i = 1 \ldots \ell \rangle.$$

In particular, Ω is finitely generated.

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- Dual cones w.r.t. standard scalar product:

$$\mathcal{V}_1^{>0} := \{x \in \mathbb{R}^n \mid xAx^{tr} < 0 \text{ and } xy_1^{tr} > 0\}$$
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- $\Omega = \mathrm{Stab}_{\mathrm{Aut}_{\mathbb{Z}}(A)}(\mathcal{V}_1^{>0})$ (*reduced* automorphism group)

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- M_ε is convex
- M_ε is not a subset of a maximal convex lattice-free set (classified by Lovász, 1989).
  $\Rightarrow \quad M_\varepsilon$ contains integral point.

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D-minimal Vectors

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- conecting elements analogously computable

Theorem (M., 2013)

Let $A \in \mathbb{Z}_{hyp}^{n\times n}$ be an integral hyperbolic matrix, Ω its reduced automorphism group and $\mathcal{V}_i^{>0}$ and D as above. Then the residue class graph Γ_D/Ω is finite. In particular, the algorithm terminates.

Watson-Process

$\Rightarrow$ smaller determinant, less perfect points

$\Rightarrow$ acceleration of computation time

Watson-Process

- $\mathrm{Aut}(L) \leq \mathrm{Aut}(\mathrm{Watson}(L))$, $[\mathrm{Aut}(\mathrm{Watson}(L)) : \mathrm{Aut}(L)] < \infty$

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Watson-Process

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- modified procedure:
  - Compute $\mathrm{Aut}(\mathrm{Watson}(L))$.
  - Find $\mathrm{Aut}(L)$ via orbit-stabilizer calculation.

$\Rightarrow$ smaller determinant, less perfect points

$\Rightarrow$ acceleration of computation time

Statistics

	∅ Time (sec.)	∅ # Points
no Watson	70.08	7.4
Watson	2.96	2.7
Isotropic	98.18	8.9
Anisotropic	2.43	5.1

Table: Computational statistics

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$$A := \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \quad A^{-1} = \frac{1}{11} \begin{pmatrix} -11 & 11 & 0 \\ 11 & -7 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

- D-perfect points:

$$x_1 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -3 & 3 & 1 \end{pmatrix},$$
$$x_3 = \begin{pmatrix} -4 & 3 & 2 \end{pmatrix}, \quad x_4 = \begin{pmatrix} -9 & 11 & 3 \end{pmatrix}.$$

- stabilizers:

$$\mathrm{Stab}(x_1) = \langle \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle \cong C_2, \quad \mathrm{Stab}(x_2) = \{1\}$$

$$\mathrm{Stab}(x_3) = \langle \begin{pmatrix} 23 & -18 & -12 \\ -8 & 5 & 4 \\ 56 & -42 & -29 \end{pmatrix}, \begin{pmatrix} 23 & -18 & -12 \\ 0 & 1 & 0 \\ 44 & -36 & -23 \end{pmatrix} \rangle \cong C_2 \times C_2,$$

$$\mathrm{Stab}(x_4) = \langle \begin{pmatrix} 5 & -8 & 0 \\ 3 & -5 & 0 \\ 1 & -2 & 1 \end{pmatrix} \rangle \cong C_2.$$

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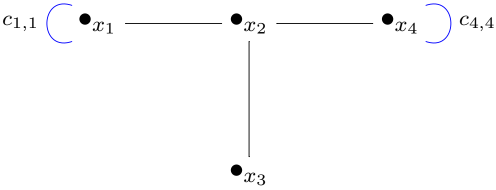
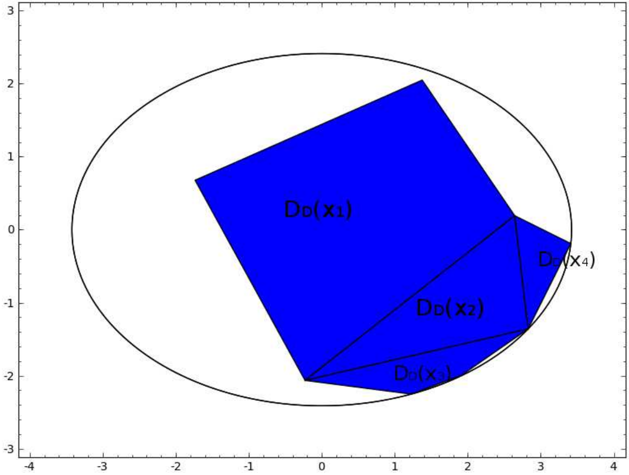
- connecting elements:

$$c_{1,1} = \begin{pmatrix} 5 & -2 & 0 \\ 3 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad c_{4,4} = \begin{pmatrix} 49 & -60 & -20 \\ 25 & -31 & -10 \\ 45 & -54 & -19 \end{pmatrix}.$$


$c_{1,1}$
x_1
x_2
x_4
$c_{4,4}$
x_3



$D_D(x_1)$
$D_D(x_2)$
$D_D(x_3)$
$D_D(x_4)$

An implementation of the algorithm in MAGMA as well as the slides of this talk are available on my homepage

http://www.mi.uni-koeln.de/∼mmertens