Moonshine beyond the Monster

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01 March, 2017
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Basic Definitions

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- $\chi : G \to \mathbb{C}$, $g \mapsto \chi(g) := \mathrm{tr}(g|V) := \mathrm{tr}(\rho(g))$ is called the character associated to ρ (or (V, ρ)).

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Theorem (I. Schur)

Let chi_i, χ_j be irreducible characters of G and $g, h \in G$. We have the following Schur orthogonality relations

$$\frac{1}{\#G} \sum_{[g]\in\mathrm{Conj}(G)} \frac{\chi_i(g)\overline{\chi_j(g)}}{\#C_G(g)} = \delta_{i,j}, \qquad \sum_{\chi\in\mathrm{Irr}(G)} \chi(g)\overline{\chi(h)} = \delta_{[g],[h]}\#C_G(g).$$

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Example

- cyclic groups C_p for p prime.
- alternating groups A_n for $n \geq 5$.
- $\mathrm{PSL}_n(\mathbb{F}_q)$ for $q \geq 4$ if $n = 2$ and all q for $n > 2$.
- finite groups of Lie type

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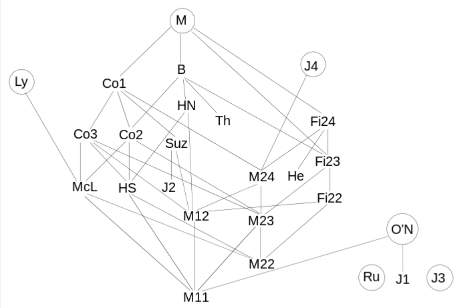
By the Jordan–Hölder Theorem, finite simple groups form the building blocks of all finite groups.

Classification of finite simple groups

Theorem

A finite simple group G either belongs to one of 8 infinite families or is one of 26 sporadic simple groups,



Source: wikipedia

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The Monster group $\mathbb{M}$

Some properties of the Monster

- The largest of the 26 sporadic finite simple groups

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- $\#\mathbb{M} = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8.08 \cdot 10^{53}$
- 194 conjugacy classes, hence 194 irreducible representations (over $\mathbb{C}$) with characters $\chi_1, \ldots, \chi_{194}$

Reminder: $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half-plane $\mathfrak{H}$ via

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) \mapsto \frac{a\tau + b}{c\tau + d}.$$

Modular forms and functions

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Definition

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup such that $\mathrm{vol}(\Gamma \setminus \mathfrak{H}) < \infty$. A meromorphic function $f : \mathfrak{H} \to \widehat{\mathbb{C}}$ is called a modular function for Γ if

$$f(\gamma\tau) = f(\tau)$$

for all $\gamma \in \Gamma$, $\tau \in \mathfrak{H}$ (+growth condition at the boundary).

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Definition

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup such that $\mathrm{vol}(\Gamma \setminus \mathfrak{H}) < \infty$. A holomorphic function $f : \mathfrak{H} \to \widehat{\mathbb{C}}$ is called a (weakly holomorphic) modular form of weight k for Γ if

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $\tau \in \mathfrak{H}$ (+growth condition at the boundary). If $\mathrm{Im}(\tau)^{\frac{k}{2}} f(\tau)$ is bounded on $\mathfrak{H}$, we call f a cusp form.

Hauptmoduln

Facts

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Definition

Let Γ be as above such that $X(\Gamma)$ has genus 0 (+ mild extra conditions). A suitably normalized generator for the field of modular functions for Γ is called the Hauptmodul for Γ.

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The Jack Daniels problem

For $N \in \mathbb{N}$ let

$$\Gamma_0(N) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \; : \; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

and

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Theorem (A. Ogg)

For p prime, the Riemann surface $X(\Gamma_0(p)^+)$ has genus zero if and only if $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$.

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Question: Why is this so?

Monstrous Moonshine I

Dimensions of irreducible representations:

$$\chi_1(1) = 1, \quad \chi_2(1) = 196\,883, \quad \chi_3(1) = 21\,296\,876, \quad \chi_4(1) = 842\,609\,326.$$

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$$J(\tau) = j(\tau) - 744 = \frac{E_4(\tau)^3}{\Delta(\tau)} - 744$$

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$$j_1 = \chi_1(1) + \chi_2(1).$$

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Values of irreducible characters at other conjugacy classes.

$$\chi_1(2A) = 1, \quad \chi_2(2A) = 4\,371, \quad \chi_3(2A) = 91\,884, \quad \chi_4(2A) = 1\,139\,374.$$

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$$J_2^+(\tau) = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 2^{12}\frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}} + 24$$
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Definition

For a finite group G let $V = \bigoplus_n V_n$ be a graded G-module, where all graded components V_n are finite-dimensional. Then for each $g \in G$ we call the power series

$$\mathcal{T}_g(q) = \sum_n \mathrm{tr}(g|V_n)q^n$$

the McKay-Thompson series of g with respect to V.

Monstrous Moonshine III

Monstrous Moonshine Conjecture

There is an infinite dimensional graded representation $V^\natural$ of $\mathbb{M}$ whose McKay-Thompson series are the 194 Hauptmoduln found by Conway–Norton.

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Theorem (R. E. Borcherds, 1992)

The Moonshine module $V^\natural$ is a vertex operator algebra constructed by Frenkel–Lepowsky–Meurman, whose automorphism group is isomorphic to $\mathbb{M}$.

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The Umbral groups

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For a Niemeier lattice L, its Umbral Group G^L is defined as

$$G^L := \mathrm{Aut}(L)/\mathsf{Weyl}(L).$$

Examples: $G^{{A_1}^{24}} = M_{24}$, $G^{{A_2}^{12}} = M_{12}$.

Umbral Moonshine I

Observation (Eguchi–Ooguri–Tachikawa, 2010)

Some dimensions of irreducible representations of M_{24} are multiplicities of superconformal algebra characters of the K3 elliptic genus, which are known to be coefficients of a (vector-valued) mock theta function.

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Theorem (T. Gannon, 2012)

There is an infinite-dimensional graded M_{24}-module whose McKay-Thompson series are specific (vector-valued) mock theta functions.

Umbral Moonshine II

Umbral Moonshine Conjecture (Cheng–Duncan–Harvey, 2012)

For every Umbral Group G^L, there is an an infinite-dimensional graded G^L-module whose McKay-Thompson series are specific (vector-valued) mock theta functions.

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Theorem (Duncan–Griffin–Ono, 2015)

The Umbral Moonshine conjecture is true.

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Dimensions of irreducible characters:

$$\chi_1(1) = 1, \quad \chi_2(1) = 248, \quad \chi_4(1) = \chi_5(1) = 27\,000,$$
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In terms of Zagier's basis of $M^{!,+}_{\frac{1}{2}}(\Gamma_0(4))$:

$$2f_3(\tau) + 248\vartheta(\tau) = 2q^{-3} + 248 + 54\,000q^4 - 171\,990q^5 + O(q^8).$$

Thompson Moonshine Conjecture

Conjecture (J. A. Harvey & B. C. Rayhaun, 2015)

There is an infinite-dimensional graded Th-supermodule $W = \bigoplus_{m\equiv 0,1\ (4)} W_m$, where $W_m = W_m^+ \oplus W_m^-$ has vanishing odd part if m is even and vice versa, whose McKay-Thompson series

$$\mathcal{T}_{[g]}(\tau) = 2q^{-3} + \sum_{\substack{m=0 \\ m\equiv 0,1\ (4)}} \mathrm{str}(g|W_m)q^m$$

are specific weakly holomorphic weight $\frac{1}{2}$ modular forms.

Main result

Theorem 1 (M. J. Griffin & M., 2016)

The Thompson Moonshine Conjecture is true. Moreover, for each $g \in Th$ there is a multiplier $\psi_{[g]}$ on $\Gamma_0(4|g|)$, s.t. $\mathcal{T}_{[g]}(\tau)$ is the unique form $\mathcal{F}_{[g]} \in M^{!,+}_{\frac{1}{2}}(\Gamma_0(4|g|), \psi_{[g]})$ with

- $\mathcal{F}_{[g]}(\tau) = 2q^{-3} + \chi_2(g) + (\chi_4(g) + \chi_5(g))q^4 + O(q^5)$,
- $\mathcal{F}_{[g]}$ has a pole of order 3 (essentially) only at the cusp ∞ and vanishes at the other cusps.

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- Theorem follows if we can show that

$$(-1)^n \omega_n = \sum_{j=1}^{48} m_j(n) \chi_j,$$

with $m_j(n) \in \mathbb{N}_0$ for all j and n.

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BUT: There is a difference between 'finite' and 'feasible'.

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Construction

Let

$$\Gamma_{K,K^2}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \ : \ |c| < K, \ |d| < K^2 \right\}.$$

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Definition

For $\mu \in \mathbb{Z}$, $k \in \frac{1}{2}\mathbb{Z}$, and ψ a multiplier system for $\Gamma_0(N)$ of weight k, we define the Rademacher sum

$$R^{[\mu]}_{\psi,k}(\tau) := \lim_{K\to\infty} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_{K,K^2}(N)} \overline{\psi}(\gamma)(q^\mu|_k\gamma).$$

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- Low-weight analogue of Poincaré series.
- Converges for $k \geq 1$, with regularization sometimes for $k < 1$.

Properties of Rademacher sums

Facts

Let $\mu < 0$.

- $R^{[\mu]}_{\psi,k}$ is a weight k mock modular form for $\Gamma_0(N)$ with multiplier ψ with shadow $R^{[-\mu]}_{\overline{\psi},2-k} \in S_{2-k}(\Gamma_0(N), \overline{\psi})$.

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"Definition"

A hol. function $f : \mathfrak{H} \to \mathbb{C}$ is called a mock modular form for $\Gamma_0(N)$ of weight k and multiplier ψ if there is a cusp form $g \in S_{2-k}(\Gamma_0(N), \overline{\psi})$ s.t.

$$\widehat{f}(\tau) := f(\tau) + \int_{-\overline{\tau}}^{\infty} \frac{\overline{g(\overline{z})}}{(z+\tau)^k} dz$$

transforms like a modular form. g is called the shadow of f, $\widehat{f}$ is the corresponding harmonic Maaß form.

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Modularity

Let

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- $\dim(S^{+}_{\frac{3}{2}}(\Gamma_0(4|g|, \overline{\psi}_{[g]})) = 0$ in most cases.

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Proof.

- $\dim(S^{+}_{\frac{3}{2}}(\Gamma_0(4|g|,\overline{\psi}_{[g]})) = 0$ in most cases.
- In other cases: $f \in S^{+}_{\frac{3}{2}}(\Gamma_0(4|g|,\overline{\psi}_{[g]})) \Rightarrow f = O(q^4)$, so the shadow of Z is 0 by the Bruinier-Funke pairing.

□

Properties of $\mathcal{F}_{[g]}$

Proposition 2

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Proof.

1. Serre-Stark basis theorem.
2. Multiply by a suitable cusp form and apply Sturm bound.

□

Integrality of multiplicities I

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- Propositions 1 & 2 yield: $\mathcal{F}_{\chi_j}$ is a weakly hol. modular form of weight $\frac{1}{2}$ of level N_{χ_j} with rational coefficients
- Checking integrality naively by Sturm bound not feasible ($N_{\chi_1} = 2\,778\,572\,160$)

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Proof.

- The $\mathcal{F}_{[g]}$ satisfy numerous congruences modulo powers of $p \mid \#Th$ (proved by Sturm bound argument, $\sim 2\,000$ coefficients to be checked at worst).
- One can then verify directly that $m_j(n)$ are p-integral for all $p \mid \#Th$, hence by Proposition 2, the claim follows.

□

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Basic strategy

Fact

Given convergence, Rademacher sums have a Fourier expansion whose coefficients are given in terms of infinite sums of Kloosterman sums

$$K_\psi(m,n,c) = \sum_{d\ (c)^*} \psi\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}\right) e^{2\pi i \frac{m\bar{d}+nd}{c}}$$

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times I-Bessel functions.

By the triangle inequality we have

$$m_j(n) \geq \frac{|\operatorname{str}(1|W_n)|}{\#Th} - \sum_{[g]\neq 1A} \frac{|\operatorname{str}(g|W_n)|}{\#C_{Th}(g)} |\chi_j(g)|.$$

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Show that from a certain point on, the first term dominates.

Positivity of the multiplicities

Proposition 4

The multiplicities $m_j(n)$ are all non-negative.

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- Check rest by inspection.

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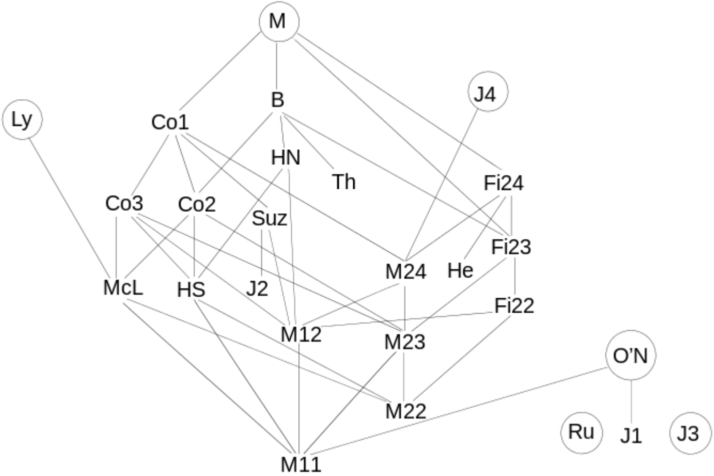
Finite simple groups



Source: wikipedia

Pariah Moonshine?

Observation

All known “Moonshine groups” so far are subquotients of $\mathbb{M}$.

Question: What about the pariahs?

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Observation (J. F. R. Duncan & M., 2016)

Zagier's weight $\frac{3}{2}$ weakly holomorphic modular form g_4:

$$-g_4 = -q^{-4} + 2 + 26\,752 q^3 + 143\,376 q^4 + 8\,288\,256 q^7 + 26\,124\,256 q^8 + O(q^{11}).$$

Some dimensions of irreducible representations of ON:

$$\chi_7(1) = 26\,752$$
$$\chi_1(1) = 1,\ \chi_{12}(1) = 58\,311,\ \chi_{18}(1) = 85\,064$$

New Moonshine

Theorem (Duncan-M.-Ono, 2017)

There is a (virtual) infinite-dimensional graded ON-module

$$W := \bigoplus_{0<m\equiv 0,3 \pmod 4} W_m$$

whose associated McKay-Thompson series are specific weight $\frac{3}{2}$ (mock) modular forms.

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Remark

We have that

$$\dim W_{163} = \frac{1}{2}(\alpha^2 + \alpha - 393768),$$

where

$$\alpha = \left\lceil e^{\pi\sqrt{163}} \right\rceil = \lceil 262537412640768743.99999999999942...\rceil.$$

ON "knows" p-parts in ideal class groups

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1. If $-D < -8$ is even and $g_2 \in \mathrm{ON}$ has order 2, then

$$\dim W_D \equiv \mathrm{tr}(g_2|W_D) \equiv -24H(D) \pmod{2^4}.$$

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2. If $p \in \{3, 5, 7\}$, $\left(\frac{-D}{p}\right) = -1$ and $g_p \in \mathrm{ON}$ has order p, then

$$\dim W_D \equiv \mathrm{tr}(g_p|W_D) \equiv \begin{cases} -24H(D) \pmod{3^2} & \text{if } p = 3, \\ -24H(D) \pmod{p} & \text{if } p = 5, 7. \end{cases}$$

ON "knows" about Selmer and Tate-Shafarevich groups I

Theorem 3 (Duncan–M.–Ono, 2017)

Assume BSD. If $p = 11$ or 19 and $-D < 0$ is a fundamental discriminant for which $\left(\frac{-D}{p}\right) = -1$, and $g_p \in \mathrm{ON}$ has order p, then the following are true.

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Let $N \in \{14, 15\}$ and write $N = p'p$, $p' < p$, and let $\delta_p := \frac{p-1}{2}$. If $-D < 0$ is a fundamental discriminant for which $\left(\frac{-D}{p}\right) = -1$, then the following are true.

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Theorem 4 (Duncan–M.–Ono, 2017)

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Thank you for your attention.