Periodicities of Taylor coefficients for half integral weight modular forms

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Expansions of modular forms

$f : \mathfrak{H} \to \mathbb{C}$ (holomorphic) modular form

"Modular forms are functions on the complex plane that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents. But they do exist." (Barry Mazur)

Expansions of modular forms

$f : \mathfrak{H} \to \mathbb{C}$ (holomorphic) modular form

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = v(\gamma)(c\tau+d)^k f(\tau),$$

$$\tau \in \mathfrak{H}, \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$$

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Modular forms have various kinds of expansions.

- Fourier expansion
- Hyperbolic expansion
- Elliptic/Taylor expansion

Fourier expansion

Some properties

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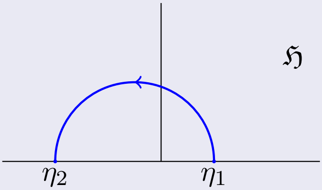
- $a_f(n)$ often encode arithmetically interesting quantities (divisor sums, number of points on elliptic curves over finite fields, partitions,...)
- congruences are well-studied ($\rightsquigarrow$ Galois representations)

Hyperbolic expansion

Some properties

- at a geodesic $\eta = \{\eta_1, \eta_2\}$



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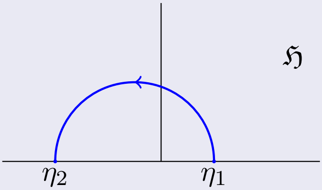

- $\mathrm{Stab}_\Gamma(\eta) = \langle \gamma_\eta \rangle \cong \mathbb{Z}$, $\gamma_\eta^{\sigma_\eta} = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$, $\xi^2 > 1$, $w = \xi^{2\tau}$;

$$(f|_k\sigma_\eta)(w) = \sum_{m\in\mathbb{Z}} b_\eta(m) w^{-k/2+\pi i m/\log\xi}$$

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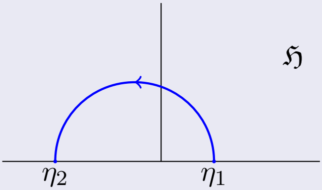

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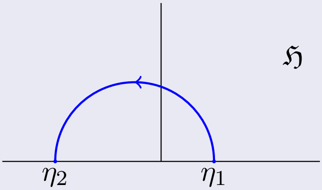

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- introduced by Petersson
- related to work by Katok, Zagier, Kohnen (holomorphic kernel of the Shimura/Shintani lift) [Imamoğlu-O'Sullivan]

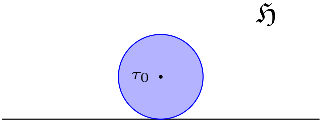
Taylor expansion I

Let $\tau_0 = x_0 + iy_0 \in \mathfrak{H}$ be an interior point.

Usual Taylor expansion

$$f(\tau) = \sum_{n=0}^{\infty} \left(\frac{d^n f}{d\tau^n}\right)(\tau_0)\frac{(\tau-\tau_0)^n}{n!}$$

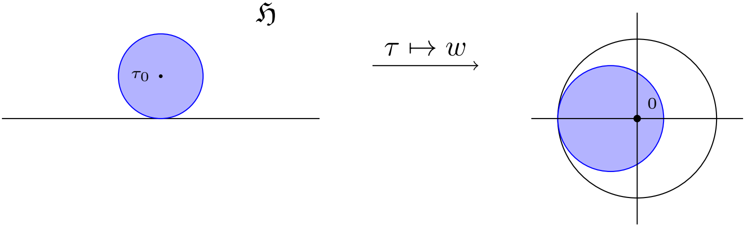
converges on $B_{y_0}(\tau_0)$



Taylor expansion I

Let $\tau_0 = x_0 + iy_0 \in \mathfrak{H}$ be an interior point.

Consider Cayley transform $\mathfrak{H} \to B_1(0),\ \tau \mapsto w = \frac{\tau - \tau_0}{\tau - \overline{\tau_0}}$ and view f as a function of w.



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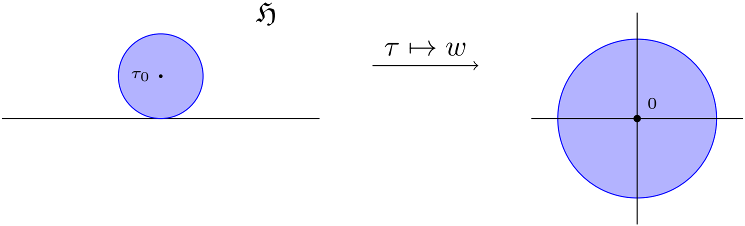

Taylor expansion II

Proposition

We have

$$(1-w)^{-k} f\left(\frac{\tau_0 - \overline{\tau_0} w}{1-w}\right) = \sum_{n=0}^{\infty} \partial^n f(\tau_0) \frac{(4\pi y_0 w)^n}{n!}, \qquad (|w| < 1),$$

where

$$\partial = \partial_k = D - \frac{k}{4\pi \operatorname{Im}(\tau)}, \quad D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq},$$

$$\partial^n = \partial_k^n = \partial_{k+2(n-1)} \circ \cdots \circ \partial_{k+2} \circ \partial_k \quad (n > 0).$$

Taylor expansion III

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- which primes are sums of two cubes? [Rodriguez-Villegas & Zagier]
- work on congruences by Larson-Smith (inert primes, integral weight, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$), Datskovsky-Guerzhoy (split primes, integral weight, $(\Gamma = \mathrm{SL}_2(\mathbb{Z}))$).

Question What arithmetic properties do Taylor coefficients of half-integral weight modular forms have?

The Jacobi theta function

Example (Romik, 2018)

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}, \qquad \tau_0 = i.$$

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$$\rightsquigarrow (1-w)^{-1/2}\theta_3\left(i\frac{1+w}{1-w}\right) = \theta_3(i)\sum_{n=0}^{\infty}\frac{d(n)}{(2n)!}\Phi^n w^{2n}, \quad |w|<1.$$

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$$d(n) = 1,\ 1,\ -1,\ 51,\ 849,\ -26199, \ldots$$

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$$d(n) \equiv \overline{1,\ 4} \pmod 5$$

$$d(n) \equiv \overline{1, 12, 12, 4, 9, 9, 3, 10, 10, 12, 1, 1, 9, 4, 4, 10, 3, 3} \pmod{13}$$

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$$d(n) \equiv 1,\ 11,\ 2,\ \overline{0} \pmod 3$$

$$d(n) \equiv 1,1,6,2,2,2,1,0,3,0,6,0,6,\overline{0} \pmod 7$$

Romik's conjecture

Conjecture (Romik, 2018)

$\{d(n)\}_{n=0}^{\infty}$ is periodic modulo $p \equiv 1 \pmod 4$ and $d(n)$ is ultimately $0 \pmod p$ for $p \equiv 3 \pmod 4$.

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Romik's conjecture is true for $p \equiv 3 \pmod 4$ and for $p = 5$.

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Question: Is this special for θ_3?

Results I

Theorem 1 (Guerzhoy-M.-Rolen, 2019)

Let $f \in M_{k-1/2}(\Gamma_1(4N))$ with algebraic integer Fourier coefficients, τ_0 a CM point, and p a prime splitting in $\mathbb{Q}(\tau_0)$. Then there exists $\Omega = \Omega(\tau_0, p) \in \mathbb{C}^\times$ such that for $n_1, n_1 > A$ with $n_1 \equiv n_2 \pmod{(p-1)p^A}$ we have

$$\partial^{n_1} f(\tau_0)/\Omega^{2k+4n_1-1} \equiv \partial^{n_2} f(\tau_0)/\Omega^{2k+4n_2-1} \pmod{p^{A+1}}.$$

Results II

Theorem 2 (Guerzhoy-M.-Rolen, 2019)

Assume $K = \mathbb{Q}(\tau_0)$ has class number 1 and the CM curve $E = \mathbb{C}/\langle \omega, \omega\tau_0 \rangle_{\mathbb{Z}}$ is defined over $\mathbb{Q}$. The there exists $\widetilde{\Omega} = \widetilde{\Omega}(\tau_0) \in \mathbb{C}^\times$ independent of p such that

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Corollary

Romik's conjecture is true.

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Quasimodular forms

Recall: $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n$ is **not** modular, but $E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi\,\mathrm{Im}(\tau)}$ is.

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Prototypical example of a quasimodular form and its associated almost holomorphic modular form.

In general: $g \in \widetilde{M}_k(\Gamma)$ can be written uniquely as

$$g = \sum_{r=0}^{\lfloor k/2 \rfloor} F_{k-2r} E_2^r \in \mathbb{C}[\![q]\!], \qquad F_{k-2r} \in M_{k-2r}(\Gamma)$$

and we have

$$g^* = \sum_{r=0}^{\lfloor k/2 \rfloor} F_{k-2r} (E_2^*)^r.$$

Damerell's theorem

Proposition 1 (Damerell, Katz)

Let K be a sufficiently large number field, $\tau_0 \in K$ be a CM point, and $k \in \mathbb{N}$. Then there exists $\omega \in \mathbb{C}^\times$ such that

$$g \in \widetilde{M}_k(\Gamma) \cap K[\![q]\!] \quad \Rightarrow g^*(\tau_0)/\omega^k \in K.$$

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Remark

- If $\omega \in \mathbb{C}^*$ works, then so daoes any K^*-multiple.
- If $\omega \in \mathbb{C}^*$ works for one $g \in \widetilde{M}_k(\Gamma) \cap K[\![q]\!]$, then it works for all such g.

q-expansion principle

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$$g_i(\tau) = \sum_{n=0}^{\infty} b_i(n) q^n \in \widetilde{M}_{k_i}(\Gamma) \cap \mathcal{O}[\![q]\!].$$

If $g_1 \equiv g_2 \pmod{p^A}$, then

$$g_1^*(\tau_0)/\omega_p^{k_1} \equiv g_2^*(\tau_0)/\omega_p^{k_2} \pmod{p^A}.$$

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N.B.: By the von Staudt-Clausen Theorem, we have $E_{p-1} \equiv 1 \pmod{p}$, so according to our mantra, we want "$E_{p-1}(\tau_0) \equiv 1 \pmod{p}$".

Sketch of proof I

Lemma

For $H \in M_k(\Gamma)$ and $G \in M_\ell(\Gamma)$ $(k, \ell \in \frac{1}{2}\mathbb{Z})$ we have $G \cdot (D^n H) \in \widetilde{M}_{k+\ell+2n}$ and $(G \cdot (D^n H))^* = G \cdot (\partial^n H)$.

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- $(\Theta D^{n_1} f)^*/\omega_p^{k+2n_1} \equiv (\Theta D^{n_2} f)^*/\omega_p^{k+2n_2} \pmod{p^{A+1}}$ (Proposition)

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- $(\Theta D^{n_1} f)^*/\omega_p^{k+2n_1} \equiv (\Theta D^{n_2} f)^*/\omega_p^{k+2n_2} \pmod{p^{A+1}}$ (Proposition)
- $\Theta(\tau_0)(\partial^{n_1} f)(\tau_0)/\omega_p^{k+2n_1} \equiv \Theta(\tau_0)(\partial^{n_2} f)(\tau_0)/\omega_p^{k+2n_2} \pmod{p^{A+1}}$ (Lemma)

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Sketch of proof II

Proof.

Proof of Theorem 2

- Need to verify that there is a global choice of $\widetilde{\Omega}$ that differs from Ω_p in Theorem 1 by a p-adic unit.

$\square$

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- Follows essentially from the fact that the Hasse invariant $A(p)$ of $E = \mathbb{C}/\langle\omega, \omega\tau_0\rangle_{\mathbb{Z}}$ over $\mathbb{F}_p$ satisfies

$$\left(\frac{2\pi i}{\omega}\right) E_{p-1}(\tau_0) \equiv A(p) \pmod{p}$$

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$$\left(\frac{2\pi i}{\omega}\right) E_{p-1}(\tau_0) \equiv A(p) \pmod{p}$$

- $A(p) \not\equiv 0 \pmod{p} \quad \Leftrightarrow \quad p$ splits in $\mathbb{Q}(\tau_0)$.

□

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Recall: $\bigoplus_k M_k(\Gamma_0(4)) = \mathbb{C}[\Theta, F_2]$ where

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}, \qquad F_2(\tau) = \frac{\eta(4\tau)^8}{\eta(2\tau)^4} = \sum_{n \text{ odd}} \sigma_1(n) q^n$$

Proposition (Guerzhoy-M.-Rolen, 2019)

Let $f \in M_k(\Gamma_0(4))$, $k \in \frac{1}{2}\mathbb{Z}$ and $P(X,Y) \in \mathbb{C}[X,Y]$ such that $f = P(\Theta, F_2)$. Then we have

$$\partial^n f(i) = \Theta(i)^{4n+2k} p_n\left((17 - 12\sqrt{2})/16\right)$$

where $p_{-1}(t) = 0$, $p_0(t) = P(X, tX^4)/X^{2k}$, and

$$p_{n+1}(t) = \frac{1}{24}(80t-1)(2k+4n)p_n(t) - (16t^2 - t)p_n'(t) - \frac{1}{144}n(n+k-1)(256t^2+224t+1)p_{n-1}(t), \qquad (n \geq 0).$$

An example I

Example

We find

$$(1-w)^{-1/2}\Theta\left(i\frac{1+w}{1-w}\right)=\Theta(i)\sum_{n=0}^{\infty}\frac{c(n)}{n!}(\Phi w)^n,\quad \Phi=\frac{(17+12\sqrt{2})\Gamma(\frac{1}{4})^4}{16\pi^2},$$

with ($\varepsilon=1+\sqrt{2}$)

n	0	1	2	3	4	5	6	7	8	9
$c(n)$	1	ε	1	-3ε	17	9ε	-111ε	2373ε	12513	86481ε

An example II

Example (continued)

Congruences:

$$\{c(n)\}_n \equiv \{1,\ \overline{\varepsilon,\ 1}^2\} \pmod 5,$$
$$\equiv \{1,\ \overline{\varepsilon, 1, 22\varepsilon, 17, 9\varepsilon, 14, 23\varepsilon, 13, 6\varepsilon, 21}^7\} \pmod{5^2},$$

and that $c(n) \equiv 57c(n+50) \pmod{5^3}$ for $n \geq 11$.
For $p = 13$, we obtain

$$\{c(n)\}_n \equiv \{1,\ \overline{\varepsilon, 1, 10\varepsilon, 4, 9\varepsilon, 6}^7\} \pmod{13}.$$

Thank you for your attention.