Mock Modular Forms and Class Number Relations

Michael H. Mertens

University of Cologne

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- M. Eichler, A. Selberg (1955/56):

$$\sum_{s\in\mathbb{Z}} (s^2-n)H(4n-s^2)+2\lambda_3(n)=0$$

$$\sum_{s\in\mathbb{Z}} (s^4-3ns^2+n^2)H(4n-s^2)+2\lambda_5(n)=0$$

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Harmonic Maaß forms

Definition

A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a harmonic weak Maaß form of weight $k \in \frac{1}{2}\mathbb{Z}$, level $N \in \mathbb{N}$, and character χ modulo N (with $4 \mid N$ if $k \notin \mathbb{Z}$) if it fulfills the following properties:

1. $(f|_k\gamma)(\tau) = \chi(d)f(\tau)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,

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2. $\Delta_k f = \left[-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right] f \equiv 0$,

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The vector space of harmonic weak Maaß forms of weight k, level N, and character χ is denoted by $\mathcal{H}_k(N, \chi)$.

Splitting

Lemma

Let f be a harmonic weak Maaß form of weight $k \neq 1$. Then f has a canonical splitting into

$$f(\tau) = f^+(\tau) + \frac{(4\pi y)^{1-k}}{k-1}\overline{c_f^-(0)} + f^-(\tau),$$

where for some $m_0, n_0 \in \mathbb{Z}$ we have the Fourier expansions

$$f^+(\tau) = \sum_{m=m_0}^{\infty} c_f^+(m) q^m$$

and

$$f^-(\tau) = \sum_{\substack{n=n_0 \\ n\neq 0}}^{\infty} \overline{c_f^-(n)} n^{k-1} \Gamma(1-k; 4\pi n y) q^{-n}.$$

The shadow

Proposition

For $k \neq 1$, the mapping

$$\xi_k : \mathcal{H}_k(N, \chi) \to M^!_{2-k}(N, \overline{\chi}),\ f \mapsto 2iy^k \overline{\frac{\partial}{\partial \overline{\tau}} f}$$

is well-defined and surjective with kernel $M^!_k(N, \chi)$. Moreover, for $f \in \mathcal{H}_k(N, \chi)$, we have that the shadow of f is given by

$$(\xi_k f)(\tau) = (4\pi)^{1-k} \sum_{n=n_0}^{\infty} c_f^-(n) q^n.$$

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For $k \neq 1$, the mapping

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$$\mathcal{M}_k(\Gamma,\chi) = \xi_k^{-1}(M_{2-k}(\Gamma,\overline{\chi}))$$
$$\mathcal{S}_k(\Gamma,\chi) = \xi_k^{-1}(S_{2-k}(\Gamma,\overline{\chi})).$$

An example

Theorem (D. Zagier, 1976)

Let $\mathscr{H}(\tau) = \sum_{n=0}^{\infty} H(n) q^n$, $\vartheta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$ and

$$\mathscr{R}(\tau) := \frac{1+i}{16\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\vartheta(z)}{(z+\tau)^{\frac{3}{2}}} dz = \frac{1}{8\pi\sqrt{y}} + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n\Gamma\left(-\frac{1}{2}; 4\pi n^2 y\right) q^{-n^2}.$$

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$$\widehat{\mathscr{H}} = \mathscr{H} + \mathscr{R} \in \mathcal{H}_{\frac{3}{2}}(4)$$

$$\xi_{\frac{3}{2}}\widehat{\mathscr{H}} = \frac{1}{8\sqrt{\pi}}\vartheta$$

Appell-Lerch sums

Definition

Let $\tau \in \mathbb{H}$, $u, v \in \mathbb{C} \setminus (\mathbb{Z} \oplus \mathbb{Z}\tau)$.
The Appell-Lerch sum of level 1 is then the expression

$$A_1(u,v;\tau) = e^{\pi i u} \sum_{n\in\mathbb{Z}} \frac{q^{\frac{n(n+1)}{2}} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n}.$$

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- the others: indefinite theta functions and Fourier coefficients of meromorphic Jacobi forms

Definition

The completion of the Appell-Lerch sum $A_1(u,v;\tau)$ is given by

$$\widehat{A}_1(u,v;\tau) = A_1(u,v;\tau) + \frac{i}{2}\Theta(v;\tau)R(u-v;\tau),$$

with

$$R(u;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left\{\operatorname{sgn}(\nu) - E\left(\left(\nu + \frac{\operatorname{Im} u}{y}\right)\sqrt{2y}\right)\right\} \times (-1)^{\nu-\frac{1}{2}} q^{-\frac{\nu^2}{2}} e^{-2\pi i\nu u}$$

$$E(t) := 2\int_0^t e^{-\pi u^2}du, \qquad \Theta(v;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} q^{\frac{\nu^2}{2}} e^{2\pi i\nu(v+\frac{1}{2})}$$

A real-analytic Jacobi form

Theorem (S. Zwegers, 2002)

$\widehat{A}_1$ transforms like a Jacobi form of weight 1 and index $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. In particular we have for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ that

$$\widehat{A}_1\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)e^{\pi i c \frac{-u^2+2uv}{c\tau+d}} \widehat{A}_1(u,v;\tau).$$

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Conjecture

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The coefficient of X^ν in

$$S_4^1(\tau, X) := \sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} \left[\sum_{s\in\mathbb{Z}} \frac{H(n-s^2)}{1-2sX+nX^2} + \sum_{k=0}^{\infty} \lambda_{2k+1}(n) X^{2k} \right] q^n$$

is a (holomorphic) modular form of weight $2\nu + 2$ on $\Gamma_0(4)$.

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- $\nu = 0$ yields Eichler's class number relation.

New class number relations

Corollary

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$$\sum_{s\in\mathbb{Z}} \left(16s^4 - 12ns^2 + n^2\right) H\left(n - s^2\right) + \lambda_5(n)$$

$$= -\frac{1}{12} \sum_{n=x^2+y^2+z^2+t^2} \left(x^4 - 6x^2y^2 + y^4\right)$$

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Cohen's conjecture is true. Moreover, $\mathrm{coeff}_{X^{2\nu}}(S_4^1(\tau; X))$ is a cusp form for $\nu > 0$.

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Remark

$$\operatorname{coeff}_{X^{2\nu}}(S_4^1(\tau;X)) = \frac{c_\nu}{2}\left([\mathscr{H},\vartheta]_\nu(\tau) - [\mathscr{H},\vartheta]_\nu\left(\tau+\tfrac{1}{2}\right)\right) + \Lambda_{2\nu+1,odd}(\tau),$$

$$c_\nu = \nu!\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})},$$

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$$\Lambda_{2\nu+1}(\tau) := \sum_{n=1}^{\infty}\lambda_k(n)q^n = \frac{1}{2}(D_v^{2\nu+1}A_1)\left(0,\tau+\frac{1}{2};2\tau\right)$$

Outline of the proof

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  - $\nu \mapsto \nu + 1$: Use heat kernel property of R ($D_u^2 R = -2D_\tau R$)

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$f : \mathbb{H} \to \mathbb{C}$ a continuous function with Fourier expansion

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Then we define the holomorphic projection of f by

$$(\pi_{hol} f)(\tau) := (\pi_{hol}^k f)(\tau) := \sum_{n=0}^{\infty} c(n) q^n,$$

with

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^\infty a_f(n, y) e^{-4\pi n y} y^{k-2} dy, \qquad n > 0.$$

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- The operator π_{hol} commutes with all the operators $U(N)$, $V(N)$, and $S_{N,r}$ (sieving operator).
- If f is modular of weight $k > 2$ on $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ then we have

$$\langle f, g \rangle = \langle \pi_{hol}(f), g \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Petersson scalar product, for every cusp form $g \in S_k(\Gamma)$.

Rankin-Cohen brackets

From now on: $f^+(\tau) + \frac{(4\pi y)^{1-k}}{k-1}\overline{c_f^-(0)} + f^-(\tau) \in \mathcal{M}_k(\Gamma)$, $g \in M_\ell(\Gamma)$, $\nu > 0$ s.t. $[f, g]_\nu$ satisfies conditions in definition.

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Lemma

$$\frac{(4\pi)^{1-k}}{k-1}\pi_{hol}([y^{1-k}, g]_\nu) = \kappa \sum_{n=0}^{\infty} n^{k+\nu-1} a_g(n) q^n,$$

with

$$\kappa = \kappa(k, \ell, \nu) = \frac{1}{(k+\ell+2\nu-2)!(k-1)} \sum_{\mu=0}^{\nu} \left[\frac{\Gamma(2-k)\Gamma(\ell+2\nu-\mu)}{\Gamma(2-k-\mu)} \times \binom{k+\nu-1}{\nu-\mu}\binom{\ell+\nu-1}{\mu} \right].$$

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Theorem

$$\pi_{hol}([f^-, g]_\nu) = \sum_{r=1}^{\infty} b(r) q^r,$$

where

$$b(r) = -\Gamma(1-k) \sum_{m-n=r} \sum_{\mu=0}^{\nu} \binom{k+\nu-1}{\nu-\mu} \binom{\ell+\nu-1}{\mu} m^{\nu-\mu} a_g(m) \overline{c_f^-(n)}$$
$$\times \left(m^{\mu-2\nu-\ell+1} P_{k+\ell+2\nu, 2-k-\mu}(r,n) - n^{k+\mu-1} \right)$$

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The Theorem of Serre-Stark

Theorem (J.-P. Serre, H. Stark, 1977)

Let φ be a modular form of weight $\frac{1}{2}$ on $\Gamma_1(N)$. Then φ is a unique linear combination of the theta series

$$\vartheta_{\chi,t}(\tau) = \sum_{n\in\mathbb{Z}} \chi(n) q^{tn^2}$$

with χ a primitive even character with conductor $F(\chi)$ and $t \in \mathbb{N}$ such that $4F(\chi)^2 t \mid N$.

Proposition

Let $r = m - n$. Then it holds that

$$\sum_{\mu=0}^{\nu} \binom{\nu+\frac{1}{2}}{\nu-\mu} \binom{\nu-\frac{1}{2}}{\mu} \left(m^{\frac{1}{2}-\nu} P_{2\nu+2,\frac{1}{2}-\mu}(r,n) - n^{\frac{1}{2}+\mu} m^{\nu-\mu} \right)$$

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$$\sum_{\mu=0}^{\nu} \binom{\nu-\frac{1}{2}}{\nu-\mu}\binom{\nu+\frac{1}{2}}{\mu}\left(m^{-\nu-\frac{1}{2}}P_{2\nu+2,\frac{3}{2}-\mu}(r,n) - n^{\mu-\frac{1}{2}}m^{\nu-\mu}\right)$$

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Proposition

Let $r = m - n$. Then it holds that

$$\sum_{\mu=0}^{\nu} \binom{\nu-\frac{1}{2}}{\nu-\mu} \binom{\nu+\frac{1}{2}}{\mu} \left(m^{-\nu-\frac{1}{2}} P_{2\nu+2,\frac{3}{2}-\mu}(r,n) - n^{\mu-\frac{1}{2}} m^{\nu-\mu} \right)$$
$$= -2^{-2\nu} \binom{2\nu}{\nu} (mn)^{-\frac{1}{2}} \left(m^{\frac{1}{2}} - n^{\frac{1}{2}} \right)^{2\nu+1}.$$

Result in weight $(\frac{3}{2}, \frac{1}{2})$

Theorem 2 (M., 2013)

Let $f \in \mathcal{M}_{\frac{3}{2}}(\Gamma)$ and $g \in M_{\frac{1}{2}}(\Gamma)$ with $\Gamma = \Gamma_1(4N)$ for some $N \in \mathbb{N}$ and fix $\nu \in \mathbb{N}$. Then there is a finite linear combination $L_\nu^{f,g}$ of functions of the form

$$\Lambda_{s,t}^{\chi,\psi}(\tau;\nu) = \sum_{r=1}^{\infty} \left(2 \sum_{\substack{sm^2 - tn^2 = r \\ m,n \geq 1}} \chi(m)\overline{\psi(n)}(\sqrt{s}m - \sqrt{t}n)^{2\nu+1} \right) q^r + \overline{\psi(0)} \sum_{r=1}^{\infty} \chi(r)(\sqrt{s}r)^{2\nu+1} q^{sr^2}$$

with $s, t \in \mathbb{N}$ and χ, ψ are even characters of conductors $F(\chi)$ and $F(\psi)$ respectively with $sF(\chi)^2, tF(\psi)^2 | N$, such that $[f,g]_\nu + L_\nu^{f,g}$ is a holomorphic modular form of weight $2\nu + 2$.

Result for mock theta functions

Theorem 3 (M., 2013)

Let $f \in \mathcal{S}_{\frac{1}{2}}(\Gamma)$ be a completed mock theta function and $g \in S^{\theta}_{\frac{3}{2}}(\Gamma)$, where $\Gamma = \Gamma_1(4N)$ for some $N \in \mathbb{N}$ and let ν be a fixed non-negative integer. Then there is a finite linear combination $D^{f,g}_{\nu}$ of functions of the form

$$\Delta^{\chi,\psi}_{s,t}(\tau;\nu) = 2\sum_{r=1}^{\infty}\left(\sum_{\substack{sm^2-tn^2=r\\ m,n\geq 1}} \chi(m)\overline{\psi(n)}(\sqrt{s}m-\sqrt{t}n)^{2\nu+1}\right)q^r,$$

where $s,t \in \mathbb{N}$ and χ, ψ are odd characters of conductors $F(\chi)$ and $F(\psi)$ respectively with $sF(\chi)^2, tF(\psi)^2 | N$, such that $[f,g]_\nu + D^{f,g}_{\nu}$ is a holomorphic modular form of weight $2\nu+2$.

Remarks

- $\Lambda_{s,t}^{\chi,\psi}$ and $\Delta_{s,t}^{\chi,\psi}$ are (up to a polynomial factor) indefinite theta functions

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Example: Eichler-Selberg and Cohen's conjecture

- Recall: $\mathscr{H}(\tau)=\sum_{n=0}^{\infty}H(n)q^n\in\mathcal{M}_{\frac{3}{2}}^{mock}(\Gamma_0(4))$ with shadow $\frac{1}{8\sqrt{\pi}}\vartheta$.

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- From Theorem 2: For $\nu>0$,

$$\pi_{hol}([\widehat{\mathscr{H}},\vartheta]_\nu)(\tau)=[\mathscr{H},\vartheta]_\nu(\tau)+2^{-2\nu-1}\binom{2\nu}{\nu}\Lambda'(\tau)\in S_{2\nu+2}(\Gamma_0(4)),$$

where $\Lambda'=\Lambda_{1,1}^{1,1}$.

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- easy to see

$$(\Lambda'|U(4))(\tau;\nu) = 2^{2\nu+1}\sum_{n=1}^{\infty} 2\lambda_{2\nu+1}(n)q^n$$

$$(\Lambda'|S_{2,1})(\tau;\nu) = 2\sum_{n \text{ odd}} \lambda_{2\nu+1}(n)q^n.$$

Example: Eichler-Selberg and Cohen's conjecture

- Rewrite:

$$\sum_{n=1}^{\infty}\left(\sum_{s\in\mathbb{Z}} g_{\nu}^{(1)}(s,n)H(4n-s^2)\right)q^n + 2\sum_{n=0}^{\infty}\lambda_{2\nu+1}(n)q^n \in S_{2\nu+2}(\Gamma_0(1))$$

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- Rankin-Selberg unfolding trick (f normalized Hecke eigenform):

$$\langle[\widehat{\mathscr{H}},\vartheta]_\nu, f\rangle = \langle\pi_{hol}([\widehat{\mathscr{H}},\vartheta]_\nu), f\rangle \doteq \langle f,f\rangle$$

$\Rightarrow$ trace formulae for $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(4)$.

Thank you for your attention.