Holomorphic Projection and Mock Modular Forms

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1. Introduction
   - Mock modular forms
   - Holomorphic projection

2. Applications
   - Construction of mock modular forms
   - Class number type relations for Fourier coefficients
   - Shifted convolution L-functions and their special values

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Ramanujan's deathbed letter

S. Ramanujan (1887-1920)

The modern definition

Definition 1

A mock modular form f of weight $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$ for $\Gamma_0(N)$ is the holomorphic part $\mathcal{M}^+$ of a harmonic Maaß form $\mathcal{M}$, i.e. there is a weakly holomorphic modular form $g \in M^!_{2-k}(\Gamma_0(N))$, the shadow of f, s.t. $\mathcal{M} = f + g^*$ with

$$g^*(\tau) := \int_{-\overline{\tau}}^{\infty} \frac{\overline{g(-\overline{z})}}{(z+\tau)^k} dz$$

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- $\Phi : \mathbb{H} \to \mathbb{C}$ continuous, transforming like a modular form of weight $k \geq 2$ for some $\Gamma_0(N)$, moderate growth at cusps (Attention for $k = 2!$).

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- same reasoning works for regularized Petersson inner product $\rightsquigarrow$ regularized holomorphic projection.

Fourier coefficients

Definition 2

If $\Phi(\tau) = \sum_{n\in\mathbb{Z}} a_\Phi(n,y)q^n$, ($y = \mathrm{Im}(\tau)$), then

$(\pi_{hol} f)(\tau) := (\pi_{hol}^{(k)} f)(\tau) := \sum_{n=0}^{\infty} c(n)q^n$, where

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!}\int_0^\infty a_\Phi(n,y)e^{-4\pi n y}y^{k-2}dy, \qquad n > 0.$$

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Remark

- For $k = 2$, $\pi_{hol}\Phi$ is a quasi-modular form of weight 2.
- For the regularized holomorphic projection, weakly holomorphic forms are possible images

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A modification of holomorphic projection

Lemma 1 (S. Zwegers)

For any translation-invariant function $\Phi : \mathbb{H} \to \mathbb{C}$ and $1 < k \in \frac{1}{2}\mathbb{Z}$ we have

$$\pi_{hol}^{(k)}(\Phi)(\tau) = \frac{(k-1)(2i)^k}{4\pi} \int_{\mathbb{H}} \frac{\Phi(z)y^k}{(\tau - \overline{z})^k} \frac{dxdy}{y^2}, \tag{1}$$

provided that the right-hand side converges absolutely.

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Lemma 2 (S. Zwegers)

Provided the rhs of (1) converges absolutely for $k \in \frac{1}{2}\mathbb{Z}$, then we have

$$(\pi_{hol}^{(k)}\Phi)|_k\gamma = \pi_{hol}^{(k)}(\Phi|_k\gamma)$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
In particular this holds if $|\Phi(\tau)|y^r$ is bounded on $\mathbb{H}$ for some r and $k > r + 1 > 1$.

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Proposition 1 (S. Zwegers)

Let Φ be as in Lemma 2. If $\pi_{hol}^{(k)}\Phi = 0$ and $\xi_k\Phi$ is modular of weight $2-k$ for some $\Gamma_0(N)$, then Φ is modular of weight k.

Surjectivity of the shadow map

Proposition (J. H. Bruinier and J. Funke)

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- multiply the Eichler integral g^* of g by a sufficiently large power of $\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24}$, say h with weight ℓ, to ensure weight and growth conditions

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- by Proposition 1, $M := \pi_{hol}^{(2-k+\ell)}(g^*h) - g^*h$ is modular of weight $2-k+\ell$ for $\Gamma_0(N)$.
- $\widetilde{M} = \frac{1}{h}M + g^*$ is the desired mock modular form.

□

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Class number relations

$$\sigma_k(n) := \sum_{d|n} d^k, \qquad \lambda_k(n) := \frac{1}{2}\sum_{d|n} \min\left(d, \frac{n}{d}\right)^k.$$

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n odd

$$\sum_{s\in\mathbb{Z}} H(n-s^2) + \lambda_1(n) = \frac{1}{3}\sigma_1(n)$$

$$\sum_{s\in\mathbb{Z}} \left(4s^2-n\right) H\left(n-s^2\right) + \lambda_3(n) = 0,$$

$$\sum_{s\in\mathbb{Z}} \left(16s^4-12ns^2+n^2\right) H\left(n-s^2\right) + \lambda_5(n)$$

$$= -\frac{1}{12}\sum_{n=x^2+y^2+z^2+t^2} \left(x^4-6x^2y^2+y^4\right),$$

...

Connection to mock modular forms

Theorem (D. Zagier)

The function

$$\mathscr{H}(\tau) := \sum_{n=0}^{\infty} H(n) q^n$$

is a mock modular form of weight $\frac{3}{2}$ for $\Gamma_0(4)$. Its shadow is (up to a constant factor) the classical theta function

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All the above relations can be formulated as

$$c_\nu[\mathscr{H}(\tau),\vartheta]_\nu|U(4) + 2\sum_{n=1}^{\infty}\lambda_{2\nu+1}(n)q^n \in \begin{cases} \widetilde{M}_2(\mathrm{SL}_2(\mathbb{Z})) & \text{if } \nu=0, \\ S_{2+2\nu}(\mathrm{SL}_2(\mathbb{Z})) & \text{if } \nu>0. \end{cases}$$

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$$\tilde{c}_\nu[\mathscr{H}(\tau), \vartheta]_\nu | S_{2,1} + \sum_{n=0}^{\infty} \lambda_{2\nu+1}(2n+1) q^{2n+1} \in \begin{cases} M_2(\Gamma_0(4)) & \text{if } \nu = 0, \\ S_{2+2\nu}(\Gamma_0(4)) & \text{if } \nu > 0. \end{cases}$$

Mock theta functions

Definition 3

A mock modular form f is called a mock theta function if its shadow is a linear combination of unary theta functions either of the form

$$\vartheta_{s,\chi}(\tau) := \sum_{n\in\mathbb{Z}} \chi(n) q^{sn^2}$$

($s \in \mathbb{N}$, χ an even character) of weight $\frac{1}{2}$ (i.e., f has weight $\frac{3}{2}$) or of the form

$$\theta_{s,\chi}(\tau) := \sum_{n\in\mathbb{Z}} \chi(n) n q^{sn^2}$$

($s \in \mathbb{N}$, χ an odd character) of weight $\frac{3}{2}$ (i.e. f has weight $\frac{1}{2}$).

Class number type relations for mock modular forms

Theorem 1 (M., 2014)

Let f be a mock theta function of weight $\kappa \in \{\frac{1}{2}, \frac{3}{2}\}$ and $g \in M_{2-\kappa}(\Gamma)$ be a l.c. of theta functions with $\Gamma = \Gamma_1(4N)$ for some $N \in \mathbb{N}$ and fix $\nu \in \mathbb{N}$. Then there is a finite linear combination $L_\nu^{f,g}$ of functions of the form

$$\Lambda_{s,t}^{\chi,\psi}(\tau;\nu) = \sum_{r=1}^{\infty} \left(2 \sum_{\substack{sm^2 - tn^2 = r \\ m,n \geq 1}} \chi(m)\overline{\psi(n)}(\sqrt{s}m - \sqrt{t}n)^{2\nu+1} \right) q^r$$

$$+\overline{\psi(0)} \sum_{r=1}^{\infty} \chi(r)(\sqrt{s}r)^{2\nu+1} q^{sr^2}$$

with $s, t \in \mathbb{N}$ and χ, ψ are characters as in Definition 3 of conductors $F(\chi)$ and $F(\psi)$ respectively with $sF(\chi)^2, tF(\psi)^2 | N$, such that $[f,g]_\nu + L_\nu^{f,g}$ is a (quasi)-modular form of weight $2\nu + 2$ (possibly weakly holomorphic).

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Notation

- Let $f_1 \in S_{k_1}(\Gamma_0(N))$ and $f_2 \in S_{k_2}(\Gamma_0(N))$ with

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- shifted convolution Dirichlet series (Hoffstein-Hulse, 2013)

$$D(f_1, f_2, h; s) := \sum_{n=1}^{\infty} \frac{a_1(n+h)\overline{a_2(n)}}{n^s}.$$

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- generating function of special values
$$\mathbb{L}^{(0)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \widehat{D}^{(0)}(f_1, f_2, h; k_1 - 1) q^h.$$

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$$\mathbb{L}^{(0)}(f_1, f_2; \tau) := \sum_{h=1}^{\infty} \widehat{D}^{(0)}(f_1, f_2, h; k_1 - 1) q^h.$$
- There is also a $\widehat{D}^{(\nu)}$ and $\mathbb{L}^{(\nu)}$ for $\nu \in \mathbb{N}_0$ (more complicated).

A numerical conundrum

$$\begin{aligned}&\mathbb{L}^{(0)}(\Delta,\Delta;\tau)\\ =&-33.383\ldots q+266.439\ldots q^2-1519.218\ldots q^3+4827.434\ldots q^4-\ldots\end{aligned}$$

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- define real numbers $\alpha = 106.10455\ldots$ and $\beta = 2.8402\ldots$, and the weight 12 weakly holomorphic modular form

$$\sum_{n=-1}^{\infty} r(n)q^n := -\Delta(\tau)(j(\tau)^2 - 1464j(\tau) - \alpha^2 + 1464\alpha),$$

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- play around a bit and find

$$-\frac{\Delta}{\beta}\left(\frac{65520}{691} + \frac{E_2}{\Delta} - \sum_{n\neq 0} r(n)n^{-11}q^n\right)$$
$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - .$$

The theorem

Theorem 2 (M.-Ono)

If $0 \leq \nu \leq \frac{k_1-k_2}{2}$, then

$$\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1-2)!} \cdot [\mathcal{M}_{f_1}^{+}, f_2]_\nu + F,$$

where $F \in \widetilde{M}^!_{2\nu+2-k_1+k_2}(\Gamma_0(N))$. Moreover, if $\mathcal{M}_{f_1}$ is good for f_2, then $F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$.

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where $F \in \widetilde{M}^!_{2\nu+2-k_1+k_2}(\Gamma_0(N))$. Moreover, if $\mathcal{M}_{f_1}$ is good for f_2, then $F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$.

- $\mathcal{M}_{f_1}$ is a harmonic Maaß form with shadow f_1. $\mathcal{M}_{f_1}$ is good for f_2 if $[\mathcal{M}_{f_1}^{+}, f_2]_\nu$ grows at most polynomially at all cusps (very rare phenomenon).

The theorem

Theorem 2 (M.-Ono)

If $0 \leq \nu \leq \frac{k_1-k_2}{2}$, then

$$\mathbb{L}^{(\nu)}(f_2, f_1; \tau) = -\frac{1}{(k_1-2)!} \cdot [\mathcal{M}_{f_1}^+, f_2]_\nu + F,$$

where $F \in \widetilde{M}^!_{2\nu+2-k_1+k_2}(\Gamma_0(N))$. Moreover, if $\mathcal{M}_{f_1}$ is good for f_2, then $F \in \widetilde{M}_{2\nu+2-k_1+k_2}(\Gamma_0(N))$.

- $\mathcal{M}_{f_1}$ is a harmonic Maaß form with shadow f_1. $\mathcal{M}_{f_1}$ is good for f_2 if $[\mathcal{M}_{f_1}^+, f_2]_\nu$ grows at most polynomially at all cusps (very rare phenomenon).
- $\widetilde{M}^!_k(\Gamma_0(N))$ is the weakly holomorphic extension of

$$\widetilde{M}_k(\Gamma_0(N)) = \begin{cases} M_k(\Gamma_0(N)) & \text{if } k \geq 4, \\ \mathbb{C}E_2 \oplus M_2(\Gamma_0(N)) & \text{if } k = 2. \end{cases}$$

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$$\mathbb{L}^{(0)}(\Delta, \Delta; \tau) = \frac{Q^+(-1, 12, 1; \tau) \cdot \Delta(\tau)}{11! \cdot \beta} - \frac{E_2(\tau)}{\beta}$$

$$= -33.383\ldots q + 266.439\ldots q^2 - 1519.218\ldots q^3 + 4827.434\ldots q^4 - \ldots.$$

Thank you for your attention.